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Fedosov observables on constant curvature manifolds and the Klein–Gordon equation

Philip C. Tillman[a,*], George A.J. Sparling[b]

[a] *Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA, USA*
[b] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, USA*

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Abstract

In this paper we construct the Fedosov star-algebra of observables on the phase–space of a single particle in the case of all (finite-dimensional) constant curvature manifolds imbeddable in a flat space with codimension one. This set of spaces includes the two-sphere and de Sitter (dS)/Anti-de Sitter (AdS) space–times. The algebra of observables was constructed by DQ techniques using, in particular, the algorithm provided by Fedosov.

The purpose of this paper was three-fold. One was to verify that DQ gave the same results as previous analyses of these spaces. Another was to verify that the formal series used in the conventional treatment converged by obtaining exact and nonperturbative results for these spaces. The last was to further develop and understand the technology of the Fedosov algorithm.

1. Introduction

Deformation quantization (DQ) yields an equivalent mathematical formulation of quantum mechanics on phase–space. The key difference between DQ and an operator formulation is that in DQ observables from classical theories are not mapped to operators—they simply stay the same. What does change or, more accurately, is introduced is a funny thing called a star-product (see [10,9]).

The star-product is simply a map $*$ that maps two functions on phase–space to another in a way that can reproduce quantum mechanics. In other words, the resulting star-algebra is isomorphic to the space of linear operators on a Hilbert space which is the usual observable algebra one works within quantum mechanics. Key relations in flat space like:

$$\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] = 0, \qquad \left[\hat{x}^{\mu}, \hat{p}_{\nu}\right] = i\hbar\delta^{\mu}_{\nu}, \qquad \left[\hat{p}_{\mu}, \hat{p}_{\nu}\right] = 0$$

are reproduced in the star-algebra as:

$$\left[x^{\mu}, x^{\nu}\right]_* = 0, \qquad \left[x^{\mu}, p_{\nu}\right]_* = i\hbar\delta^{\mu}_{\nu}, \qquad \left[p_{\mu}, p_{\nu}\right]_* = 0$$

* Corresponding address: Department of Physics and Astronomy, University of Pittsburgh, 225 Old Engineering Hall3941 O'Hara St., 15260 Pittsburgh, PA, USA. Tel.:+1 412 624 8702; fax: +1 412 624 9163.
E-mail addresses: phil.tillman@gmail.com (P.C. Tillman), sparling@twistor.org (G.A.J. Sparling).

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where the commutator $[f, g]_* = f * g - g * f$ for all phase–space functions f and g. Also, the star-product is associative and linear as is dictated by quantum mechanics and the presence of Hilbert space representations.

Fedosov [2] has provided an algorithm to construct a star-product as a formal series in $\hbar$ on any finite-dimensional symplectic manifold. The algorithm's power is that it is geometrical and does not rely on coordinate dependent things. To understand the basic idea of the algorithm of Fedosov we should understand the Groenewold–Moyal star-product.

The birth of Moyal star-product (hence the birth of DQ) relies on the quantization map given by the Weyl quantization map (usually written as an integral transform) $\mathcal{W}$. The Weyl quantization map assigns to each phase–space function a unique observable by symmetric ordering, for example:

$$\mathcal{W}\left(x^2 p\right) = \frac{1}{3}\left(\hat{x}^2\hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2\right)$$

in general we have $\mathcal{W}(ax+bp)^n = \left(a\hat{x} + b\hat{p}\right)^n$. Now, we can use Wigner's inverse map $\mathcal{W}^{-1}$ (the inverse of the integral transform) and we can find the Groenewold–Moyal star-product defined as:

$$f * g := \mathcal{W}^{-1}\left(\mathcal{W}(f)\,\mathcal{W}(g)\right).$$

Groenewold [7] (and later Moyal [11]) investigated this formula and found a remarkable result:

$$f * g = f \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right] g.$$

In a coordinate independent formulation we have:

$$\begin{aligned} f * g &= f \exp\left[(i\hbar/2)\overleftrightarrow{P}\right] g \\ &= \sum_{A,B,j}^{\infty} (i\hbar/2)^j\, \omega^{A_1 B_1}\cdots\omega^{A_j B_j}/j!(\partial_{A_1}\cdots\partial_{A_j} f)(\partial_{B_1}\cdots\partial_{B_j} g) \\ \overleftrightarrow{P} &:= \overleftarrow{\partial}_A \omega_{AB} \overrightarrow{\partial}_B \end{aligned} \tag{1.1}$$

where $\overleftrightarrow{P}$ is the Poisson bracket and ∂_A is a (flat) torsion-free phase–space connection ($\partial \otimes \omega = 0$). Also, $q^A = \left(x^\mu, p_\mu\right)$ and $\partial_A q^B = \delta_A^B$. The capital Latin indices A, B, etc. are numerical phase–space indices and run from 1 to $2n$ while the Greek lower-case indices represent numerical space–time indices. We will sometimes use abstract space–time indices represented by lower-case Latin letters.

The Fedosov algorithm does the same basic thing, first find the quantization map he calls σ^{-1} then use its inverse to define the Fedosov star-product by:

$$f * g := \sigma\left(\sigma^{-1}(f)\,\sigma^{-1}(g)\right).$$

In this paper we construct the Fedosov star-product on an arbitrary (finite-dimensional) constant curvature manifold of codimension one (the precise definition of this manifold will be given later) by constructing the map σ^{-1}. This paper is a straightforward generalization of our previous paper [12], where in that paper we considered the two-sphere case of which the case considered now subsumes. Along the way we will derive some formulas (and some properties thereof) that are completely general for a finite-dimensional phase–space of a configuration space which represents our space–time. We feel that they may be useful for future calculations of the Fedosov star-product.

What is shown is that following an *exact and nonperturbative* calculation, the resulting star-algebra is the pseudo-orthogonal group $\mathbb{SO}(p+1, q+1)$ where p and q are fixed by the embedding formula. This also addresses the question of convergence of the map σ^{-1} which is a critical problem of the general Fedosov star and DQ in general. Also, the Klein–Gordon equation is given by a Casimir invariant of a subgroup, either $\mathbb{SO}(p, q+1)$ or $\mathbb{SO}(p+1, q)$ (the choice again depends on the embedding formula). We note that the subgroup $\mathbb{SO}(p, q+1)$ or $\mathbb{SO}(p+1, q)$ is the symmetry group of this constant curvature manifold. These results are completely expected and consistent with the analysis of Frønsdal [3–6] which is the standard theory of particles on de Sitter (dS) and Anti-de Sitter (AdS) space–times in $1+3$ dimensions. The advantage of our result using the Fedosov star-product is that it is algorithmic whereas the results achieved by Frønsdal and others rely, crucially, on symmetries of the particular case considered.

1.1. Outline

Before the readers begin this paper, they should familiarize themselves with the notations in Appendix A. In Section 2 the Fedosov star-product is defined by means of its algorithm. The properties as well as how to formulate the Klein–Gordon equation in general in DQ are discussed.

Section 3 states the original results of this paper. Beginning with the background geometry and a phase–space connection we construct the Fedosov star for the phase–space of any constant curvature manifold of codimension one. This class of manifolds include the two-sphere, dS, and AdS. The background geometry is reviewed as well as a phase–space connection is introduced.

2. The Fedosov star-product

On a flat phase–space the Weyl quantization map $\mathcal{W}$ is the isomorphism between the algebra of observables on a Hilbert space and the Groenewold–Moyal star-algebra on phase–space. The goal of the Fedosov algorithm is to construct a similar map called σ^{-1} on a general phase–space which associates a unique Hilbert space operator $\hat{f}$ to each phase–space function f. The map σ^{-1} in [2] is a flat section in the Weyl–Heisenberg bundle (something which we will define later). The star-product of any two phase–space functions would be defined by:

$$f * g := \sigma\left(\sigma^{-1}(f)\,\sigma^{-1}(g)\right)$$

analogously to the definition of the Groenewold–Moyal star-product (1.1). Fedosov provides an algorithm (see [2]) to construct the map σ^{-1} and σ. However, the construction of such a map is a non-trivial task as we will see in the following sections. With convergence issues aside, the properties of the Fedosov star are (see [2,12]):

1. It is coordinate independent.
2. It can be constructed on all symplectic manifolds (including all phase–spaces) perturbatively in powers of $\hbar$.
3. It assumes no dynamics (e.g. Hamiltonian or Lagrangian), symmetries, or even a metric.
4. The limit $\hbar \to 0$ yields classical mechanics.
5. It is equivalent to an operator formalism by a Weyl-like quantization map σ^{-1}.

In this paper we will restrict the focus onto phase–spaces of finite-dimensional manifolds because these are the most relevant for the type of physics we are interested in.

Definition. A **symplectic manifold** is manifold equipped with a nondegenerate (i.e., at all points ω_{AB} has an inverse ω^{AB} st. $\omega^{AB}\omega_{BC} = \delta^A_C$) closed two-form.

It is well-known that all phase–spaces are symplectic manifolds. Consider $T^*\mathbb{R}^n$, the phase–space of $\mathbb{R}^n$. Choose the coordinates of the configuration space $\mathbb{R}^n$ to be x^μ then there exists canonical momentum associated to these coordinates p_μ. In these coordinates of phase–space (x, p) the symplectic form is $\omega = dp_\mu dx^\mu = dp_1 dx^1 + \cdots + dp_n dx^n$. The Poisson bracket is then $\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\mu}$.

Definition. The **cotangent space** $T^*_x M$ of a manifold M at the point $x \in M$ is the vector space of all possible momenta p_μ.

Definition. The **cotangent bundle** or **phase–space** of M is $T^*M = \cup_{x\in M} T^*_x M$ of a manifold M is the union of all tangent spaces at all points $x \in M$. A point in this space is represented by (x, p).

For this paper let M be space–time. It is a fact for any M that T^*M is always equipped with a nondegenerate closed two-form ω which is basically the inverse of the Poisson bracket tensor.[1] This is a straightforward generalization of the above example in $\mathbb{R}^n$ because we always have canonical momenta associated to each choice of coordinates x^μ. Therefore, every phase–space is a symplectic manifold. The symplectic form in some local coordinates (x, p) is $\omega = dp_\mu dx^\mu$ where x is the coordinate on M and p is the canonical momentum conjugate to x. Also, on every

[1] The Poisson bracket tensor has two upstair indices and so is a (2, 0) tensor and the symplectic form is a (0, 2) tensor.

phase–space we can define a phase–space connection D which we will need for the construction of the Fedosov star-product. We define the Fedosov triple by (T^*M, ω, D).

For any Fedosov triple, Fedosov gives a perturbative expansion for a generalized Groenewold–Moyal star-product, the so-called Fedosov star-product. We note here that since the star-product is formulated in terms of a perturbative expansion its convergence issues remain unknown in general.

2.1. The Klein–Gordon (KG) equation on an arbitrary space–time

In order to gain a basic feel for this new formulation of quantum mechanics we should re-express the fundamental quantities and equations into it. Here we express the Klein–Gordon equation into this new language, i.e., into DQ. In Minkowski space this is done by the use of the isomorphism of the Weyl quantization map $\mathcal{W}$.

In special relativistic mechanics on Minkowski space the quantization of a single particle begins with the classical invariant:

$$p_\mu p^\mu - m^2 = 0.$$

This invariant is then promoted to a constraint on the set of physically allowed states where m is the rest mass of the particle. The resulting equation is the eigenvalue equation:

$$\left(\hat{p}_\mu \hat{p}^\mu - m^2\right)|\phi_m\rangle = 0, \qquad \langle\phi_m|\phi_m\rangle = 1$$

and computing:

$$\hat{H}\hat{\rho}_m = \hat{\rho}_m \hat{H} = m^2\hat{\rho}_m, \qquad \mathrm{Tr}\left(\hat{\rho}_m\right) = 1, \qquad \hat{\rho}_m^\dagger = \hat{\rho}_m, \qquad \hat{\rho}_m^2 = \hat{\rho}_m$$

where $\hat{H} = \hat{p}_\mu \hat{p}^\mu$, $\hat{\rho}_m := |\phi_m\rangle\langle\phi_m|$, Tr is the full trace, and $|\phi_m\rangle$ is the state of a spin zero particle.

This equation can then be mapped to phase–space by $\mathcal{W}^{-1}$:

$$H * \rho_m = \rho_m * H = m^2\rho_m\ , \qquad \mathrm{Tr}_*\left(\rho_m\right) = 1, \qquad \bar{\rho}_m = \rho_m, \qquad \rho_m * \rho_m = \rho_m \tag{2.1}$$

$$H = p_\mu * p^\mu \tag{2.2}$$

where $*$ is the Groenewold–Moyal star-product, $g_{\mu\nu}(x)$ is the configuration space metric, $H = p_\mu p^\mu$ $(p^\mu := g^{\mu\nu} p_\nu)$ and ρ_m is the function that represents an eigenstate of H.

In an analogous derivation (and by adding an arbitrary Ricci term[2]). We can formulate the KG equation on an arbitrary space–time in DQ using the map σ^{-1} provided by Fedosov's algorithm. H is now replaced with a new $H = p_\mu * p^\mu + \xi R$ where $R = R(x)$ is the Ricci curvature scalar associated to this metric $g_{\mu\nu}(x)$ of the space–time, $\xi \in \mathbb{C}$ is an arbitrary constant, and $*$ is now the Fedosov star-product.

The equation:

$$\left(\hat{p}_\mu \hat{p}^\mu + \xi\hat{R} - m^2\right)|\phi_m\rangle = 0, \qquad \langle\phi_m|\phi_m\rangle = 1 \tag{2.3}$$

becomes:

$$H * \rho_m = \rho_m * H = m^2\rho_m, \qquad \mathrm{Tr}_*\left(\rho_m\right) = 1, \qquad \bar{\rho}_m = \rho_m, \qquad \rho_m * \rho_m = \rho_m \tag{2.4}$$

$$H = p_\mu * p^\mu + \xi R. \tag{2.5}$$

2.2. The algorithm

In this section we provide a brief outline of the algorithm that is used to construct the Fedosov star-product. Since some of the formulas are put into more convenient forms, constraints are carried through, as well as many other complications, we want to illustrate what the algorithm does.

2 The reason we add an arbitrary Ricci term is that we cannot disallow it. This term is standard in many texts like [1].

Step 1. We begin with a phase–space connection D:

$$Df = df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial p_\mu} dp_\mu$$
$$D \otimes \Theta^A = \Gamma^A_B \otimes \Theta^B = \Gamma^A_{BC} \Theta^C \otimes \Theta^B$$

where Θ^A is a basis of one-forms in the cotangent bundle of our phase–space (for example let $\Theta^A = (dx^\mu, dp_\mu)$). The symbol Γ^A_{BC} is defined to be the Christoffel symbol. The connection preserves the symplectic two-form $\omega = \omega_{AB} \Theta^A \wedge \Theta^B$ (the inverse of the Poisson bracket tensor ω^{AB}, i.e., $\omega^{AB} \omega_{BC} = \delta^A_C$) by $D \otimes \omega = 0$. In the coordinates $\left(x^\mu, p_\mu\right) \omega = dp_\mu \wedge dx^\mu$. The Poisson bracket operator is $\omega^{AB} \frac{\partial}{\partial q^A} \wedge \frac{\partial}{\partial q^B}$.

Step 2. To each point $q = (x, p)$ on the phase–space we associate a matrix algebra called the Heisenberg–Weyl algebra. The union of these algebras is called the Weyl–Heisenberg bundle over the phase–space. We define the basis elements $\hat{y}^A$ as an infinite-dimensional matrix. $\hat{y}^A$ is defined to have the properties:

$$\left[\hat{y}^A, \hat{y}^B\right] = \hat{y}^B \hat{y}^A - \hat{y}^B \hat{y}^A = i\hbar \omega^{AB} \hat{1} \tag{2.6}$$

$$D\hat{y}^A = \Gamma^A_{BC} \Theta^C \hat{y}^B \tag{2.7}$$

where $\hat{1}$ is the identity matrix and it is assumed that Θ are treated as a scalar with respect to $\hat{y}$'s matrix indices ($[\Theta^A, \hat{y}^B] = 0$).

*Note that we will omit the $\hat{1}$ from the formula from now on and it is implicitly there.

To better understand these $\hat{y}^A$ we should think of them as a matrix with matrix-elements which are functions. Explicitly we have:

$$\hat{y}^A = \begin{pmatrix} y^A_{11}(x, p) & y^A_{12}(x, p) & \cdots \\ y^A_{21}(x, p) & y^A_{22}(x, p) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

so that $y^A_{ij}(x, p)$ is a function for each i and j.

Step 3. We define a matrix operator called $\hat{D}$ defined by the graded commutator[3]:

$$\hat{D} = \left[\hat{Q}, \cdot\right] / i\hbar = \left[\hat{Q}_A \Theta^A, \cdot\right] / i\hbar$$
$$\hat{Q}_A = \sum_l Q_{AA_1 \cdots A_l} \hat{y}^{A_1} \cdots \hat{y}^{A_l}$$

where $Q_{AA_1 \cdots A_l}$ are complex-valued functions of x and p that need to be determined.

The coefficients $Q_{AA_1 \cdots A_l}$ are partially determined[4] by the condition:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = 0. \tag{2.8}$$

We can fix $\hat{D}$ any way we like, just as long as the above condition holds. The way to think of the above condition is as an integrability condition in the construction of the observable algebra.

Step 4. We then use $\hat{D}$ to define the algebra of observables to be the set of all functions $\hat{f}$:

$$\hat{f}\left(x, p, \hat{y}\right) = \sum_{j,l} f_{j,l,A_1 \cdots A_l} \hbar^j \, \hat{y}^{A_1} \cdots \hat{y}^{A_l} \tag{2.9}$$

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[3] Graded commutators have the property that $\left[\hat{Q}_A \Theta^A, w\right] = \left[\hat{Q}_A, w\right] \Theta^A = \left(\hat{Q}_A w - w\hat{Q}_A\right) \Theta^A$ where w is an arbitrary l-form with coefficients $w_{A_1 \cdots A_l}$ which are complex-valued functions of the variables x, p and $\hat{y}$.

[4] Fedosov adds an additional condition that makes his $\hat{D}$ unique from a fixed D being $\hat{d}^{-1} r_0 = 0$ where $\hat{d}^{-1}$ is what he calls δ^{-1} (an operator used in a de Rham decomposition) and r_0 is the first term in the recursive solution. We regard this choice as being artificial and thus omit it from the paper.

where $f_{j,l,A_1\cdots A_l}$ are complex-valued functions of x and p for each $j, l, A_1, \ldots, A_l$ that need to be determined. Moreover the indices $(A_1 \cdots A_l)$ are assumed to be symmetric.

For every function $f(x, p)$ the coefficients $f_{j,l,A_1\cdots A_l}$ of the series above are partially determined by the conditions:

$$\left(D - \hat{D}\right) \hat{f} = 0 \qquad (2.10)$$
$$\sigma(\hat{f}) = f_{0,0} = f(x, p)$$

where σ is defined to be:

$$\sigma(\hat{f}) = \sum_{j,l} f_{j,0}\, \hbar^j .$$

We can make any choice that fixes the additional freedom and in total this gives us the map we need σ^{-1}.

Note: The inverse quantization map σ is defined to be the operation that picks out the leading order term in the symmetrized series for f in (2.9), i.e., the term that has no $\hat{y}$'s in them.

Step 5. The Fedosov star-product $f * g$ is defined by:

$$f * g := \sigma(\sigma^{-1}(f)\,\sigma^{-1}(g)) = \sigma(\hat{f}\hat{g}).$$

*Note that to get the leading order term $\sigma(\hat{f}\hat{g})$ you have to symmetrize all the monomials in $\hat{y}$'s in the product $\hat{f}\hat{g}$ first, then take the leading term. This makes the multiplication of $f * g$ highly non-trivial.

There is some freedom in choosing D and $\hat{D}$ but once they are chosen we can associate unique operators $\hat{f}$ to every phase–space function. This is precisely the map that we need σ^{-1} (and σ), $\sigma^{-1}(f) = \hat{f}$ (i.e., σ^{-1} is a section in the bundle). Moreover, the reason we call σ^{-1} a flat section because it is constructed with the condition that the curvature of the derivation $(D - \hat{D})$ is zero, the condition (2.8).

3. The Fedosov star-product on constant curvature manifolds of codimension one

Now that we are familiar with the basics of DQ and the Fedosov star-product, we shall explicate the results of the paper. The focus of this paper is on a particular star-product known as the Fedosov star-product. Fedosov star-product is a star-product that can be written down at least in a formal power series in $\hbar$ for any generalization of a phase–space of arbitrary space–time manifold called a symplectic manifold. Although symplectic manifolds are more general manifolds than phase–spaces, we will only consider phase–spaces.

As stated before, the primary aim of the paper was to construct the Fedosov star-product on the phase–space of a single particle in the case of all (finite-dimensional) constant curvature manifolds embeddable in a flat space with codimension one. The observable algebra is algebra of functions on phase–space along with this new product. This set of spaces includes the two-sphere and de Sitter (dS)/Anti-de Sitter (AdS) space–times. By techniques provided by Fedosov's algorithm we can construct the quantization map σ^{-1} (and also σ but this map is trivial to construct so all of our hard work goes into σ^{-1}) which is what these results do. The crucial ingredient is the construction of a new derivation $\hat{D}$ so that $(D - \hat{D})$ is a flat derivation in step 3. This derivation is crucial to the definition of the algebra in step 4. From this we can write down the star-product for any phase–space functions in powers of $\hbar$.

The purpose of these results was four-fold. One was to verify that DQ gave the same results as the previous analyses of these spaces. Another was to verify that the formal series obtained by the Fedosov algorithm converged by obtaining *exact* and *nonperturbative* results for these spaces. As was stated in the introduction, one the most serious issues confronting DQ is the issue of convergence of all formal series in $\hbar$. Therefore, if the star-product has any merit at all in describing quantum theories on non-trivial manifolds it should be well-defined for some of the simplest cases, i.e., constant curvature manifolds.

The last goal was to further develop the technology of the Fedosov algorithm. This includes developing a refinement of the formulas for the algorithm by assuming that the symplectic manifold is a phase–space. We then show that the resulting condition (3.24) is locally integrable by the Cauchy–Kovalevskaya theorem.

This section will read as follows: Each subsection (excluding the background geometry subsection and the last three subsections of this section) will remain completely general for an arbitrary phase–space until the sub-subsection entitled: "The Constant Curvature Case Explicitly". It is here we will state results specifically for the constant curvature manifold case of codimension one. It is in the part of the subsection preceding this that we will derive some general formulas and so will be valid for all finite-dimensional phase–spaces.

3.1. The background geometry

Before we go into the details of the results we first want to review the geometry of constant curvature manifolds of codimension one. To this end, we rely on the fact that it is a relatively straightforward generalization of the familiar two-sphere and dS/AdS manifolds. The fact that the sphere and dS/AdS lie in this class is the main motivation for considering it.

We start with the phase space of a single classical particle confined to a constant curvature manifold with metric $\left(M_{C_{p,q}}, g\right)$ that is imbedded in $\left(\mathbb{R}^{n+1}, \eta\right)$ where $\dim M_{C_{p,q}} = p + q = n$ and η is a pseudo-euclidean metric. The imbedding specifically is the hyperboloid:

$$x^{\mu} x_{\mu} = \eta_{\mu\nu} x^{\mu} x^{\nu} = 1/C$$

η induces a metric on $M_{C_{p,q}}$ called g and explicitly:

$$g_{\mu\nu} := \eta_{\mu\nu} - C x_{\mu} x_{\nu} \tag{3.1}$$

which is easily obtained by the constraint above (just project each index orthogonal to x). Also, we will always raise and lower the lower-case indices or $M_{C_{p,q}}$ indices (Greek or Latin) by the metric of the imbedding space $\mathbb{R}^{n+1}\eta$.

We make the convention that the positive signature directions are the "time" directions while the negative ones are the "space" directions. If the signature of g denoted by $\operatorname{sign}(g)$ is (p, q) then for $C > 0$ (this space–time is denoted by $M^{+}_{C_{p,q}}$), η is a pseudo-euclidean metric of signature $(p + 1, q)$ or explicitly:

$$\eta = \operatorname{diag}\left(\underbrace{1, \ldots, 1}_{p+1}, \underbrace{-1, \ldots, -1}_{q}\right).$$

If, however, $C < 0$ (this space–time is denoted by $M^{-}_{C_{p,q}}$), η is a pseudo-euclidean metric of signature $(p, q + 1)$. This is because for $C > 0$ the hyperboloid is "time"-like, i.e., it has normal vectors pointing in a combination of the $p + 1$ positive signature directions thus the induced metric has a signature of one less "time" dimensions from the imbedding. For the case of $C < 0$ the hyperboloid is space-like and thus the induced metric has a signature of one less "space" dimensions, i.e., it has normal vectors pointing in a combination of the $q + 1$ negative signature directions.

A good way to visualize these spaces is to look at the $1 + 3$ dimensions which gives us the familiar de Sitter (dS) and Anti-de Sitter (AdS) space–times for $C < 0$ and $C > 0$ respectively. The picture, of course, generalizes very naturally. The embeddings in these cases are:

$$\left(x^0\right)^2 - \left(x^4\right)^2 - \underline{x} \cdot \underline{x} = 1/C, \quad C < 0 \tag{3.2}$$

$$\left(x^0\right)^2 + \left(x^4\right)^2 - \underline{x} \cdot \underline{x} = 1/C, \quad C > 0 \tag{3.3}$$

where:

$$\underline{x} = \left(x^1, x^2, x^3\right).$$

The picture of dS is in Fig. 3.1 and AdS is in Fig. 3.2.

We notice that in the case of dS the definition of time must be x^0 and in AdS it must be the 0–4 angle θ. We immediately notice a problem in this embedding of AdS: If we follow a world line starting at $\theta = 0$ and ending at $\theta = 2\pi$ we arrive back at our starting point. We reason that we cannot reach the past by going far into the future. This is to avoid serious paradoxes of what must be a pathological space–time.

The resolution to this dilemma is to go to the covering space of the hyperboloid by "unidentifying" (or not identifying them in the first place) the values $0, \pm 2\pi, \pm 4\pi, \ldots$. This is done by breaking the hyperboloid into leaves (labelled by n) and so if we follow a world-line starting at $\theta = 0$ when we get to 2π we will be in a different

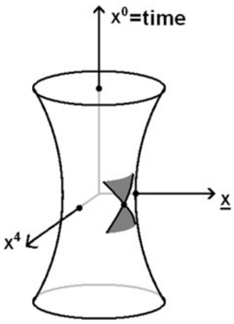

Fig. 3.1. dS space–time ($C < 0$). An arbitrary light cone is drawn with the shaded region being the time-like direction.

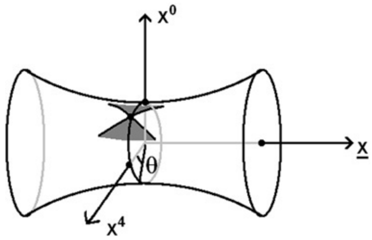

Fig. 3.2. AdS space–time ($C > 0$). An arbitrary light cone is drawn with the shaded region being the time-like direction.

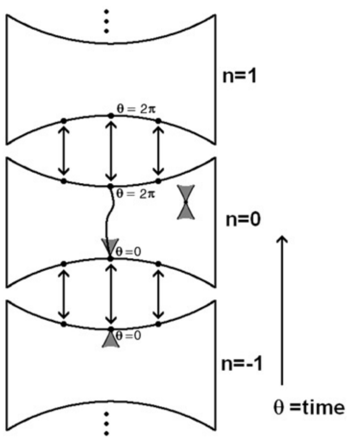

Fig. 3.3. The covering space of dS space–time. Here the observer who follows the world-line starting at $\theta = 0$ and ending at $\theta = 2\pi$ does not arrive back in his past. An arbitrary light cone is drawn with the shaded region being the time-like direction.

leaf of the covering space and thus not at our original point. The picture is described by first imagining that we have infinitely many hyperboloids. We then cut them length-wise, open them up, and put each successive one above the other as follows is seen in Fig. 3.3. Thus the topology of time is $\mathbb{R}$ not an $\mathbb{S}^1$.

By differentiating $x^\mu x_\mu = 1/C$ we may obtain the condition on p_μ:

$$2dx^\mu x_\mu = 0 \Longrightarrow x^\mu p_\mu = 0.$$

The embedding formulas are then:

$$x^\mu x_\mu = 1/C, \quad x^\mu p_\mu = 0 \tag{3.4}$$

where C is an arbitrary real constant.

3.2. The phase–space connection

The starting place of the Fedosov algorithm is the phase–space connection D in step 1 of the algorithm. In this section we construct a connection suitable for our purposes although any could be chosen. We choose a torsion-free phase–space connection that preserves the metric. To construct D we start with the Levi-Civita connection ∇ on the configuration space M and use this to derive the desired phase–space connection.

We now introduce a Levi-Civita connection ∇ on the configuration space M and subsequent curvature given the metric g on a general manifold M:

$$\begin{aligned} &\nabla_\sigma f(x) = \frac{\partial f}{\partial x^\sigma} \\ &\nabla_\sigma \left(dx^\mu\right) = -\Gamma^\mu_{\nu\sigma} dx^\nu \\ &\nabla_\sigma \left(\frac{\partial}{\partial x^\mu}\right) = \Gamma^\nu_{\mu\sigma} \frac{\partial}{\partial x^\nu} \\ &\nabla_{[\sigma}\nabla_{\rho]} \left(dx^\mu\right) = R^\mu_{\ \nu\sigma\rho} dx^\nu \end{aligned} \tag{3.5}$$

where $R^\mu_{\ \nu\sigma\rho}$ is the Riemann tensor. Of course we have the conditions that ∇ preserves the metric g and is torsion-free:

$$\begin{aligned} &\nabla_a g_{bc} = 0 \\ &\nabla_{[a}\nabla_{b]} f(x) = 0 \end{aligned}$$

for all functions $f(x)$. Together these uniquely fix ∇.

We can "lift" the action of this connection ∇ to induce a unique phase–space connection D (see Appendix D for the details). The way we can think of this induction is that the configuration space connection ∇ acts naturally on the covectors of covectors (which are essentially two-index tensors). On the cotangent bundle of phase–space we define a basis of one-forms (dx^μ, α_μ) where α_μ is defined as:

$$\alpha_\mu := dp_\mu - \Gamma^\nu_{\mu\rho} dx^\rho p_\nu. \tag{3.6}$$

We define the phase–space connection to be:

$$\begin{aligned} &Dx^\mu := dx^\mu \\ &Dp_\mu := dp_\mu \\ &D \otimes dx^\mu = -\Gamma^\mu_{\sigma\nu} dx^\nu \otimes dx^\sigma \\ &D \otimes \alpha_\mu = \Theta^B \otimes D_B \alpha_\mu := -\frac{4}{3} R^\psi_{\ (\mu\sigma)\beta} p_\psi dx^\beta \otimes dx^\sigma + \Gamma^\nu_{\mu\sigma} dx^\sigma \otimes \alpha_\nu \\ &\alpha_\mu := dp_\mu - \Gamma^\nu_{\mu\rho} dx^\rho p_\nu \end{aligned} \tag{3.7}$$

and the corresponding curvature:

$$\begin{aligned} &D^2 x^\mu = 0 \\ &D^2 p_\mu = 0 \\ &D^2 \otimes dx^\mu = dx^\sigma dx^\rho \otimes R^\mu_{\ \nu\sigma\rho} dx^\nu \\ &D^2 \otimes \alpha_\mu = \frac{4}{3} dx^\sigma \left(C^\psi_{\ \mu\beta\nu\sigma} p_\psi dx^\nu + R^\nu_{\ (\mu\beta)\sigma} \alpha_\nu\right) \otimes dx^\beta - R^\nu_{\ \mu\sigma\beta} dx^\sigma dx^\beta \otimes \alpha_\nu \end{aligned} \tag{3.8}$$

where $C^c_{\ abes} := \nabla_s R^c_{\ (ab)e}$ and according to (3.5) the formula for the curvature is:

$$R^\mu_{\ \nu\sigma\rho} = -\partial_{[\sigma} \Gamma^\mu_{\ \rho]\nu} + \Gamma^\kappa_{\ \nu[\sigma} \Gamma^\mu_{\ \rho]\kappa}. \tag{3.9}$$

We can extend to higher-order tensors by using the Leibnitz rule and the fact that D and ∇ commute with contractions.

3.2.1. The constant curvature case explicitly

Given a configuration space connection ∇ it was a relatively straightforward matter to derive a phase–space connection associated to it. So all we need is formulas for the Christoffel symbols Γ in our coordinates and we are done. Normally this would be a straightforward matter, but because of the constraints:

$$x^{\mu}x_{\mu} = 1/C\ , \qquad x^{\mu}p_{\mu} = 0 \tag{3.10}$$

and the subsequent conditions:

$$x^{\mu}dx_{\mu} = 0\ , \qquad p_{\mu}dx^{\mu} + x^{\mu}dp_{\mu} = 0 \tag{3.11}$$

the situation becomes a bit more muddled.

Without constraints when given a metric the Levi-Civita (torsion-free and metric-preserving) and its curvature would be determined uniquely by the formulas (D.2) and (3.9) in Appendix D. However, when we compute them using these formulas we are still left with freedom resulting from the above constraint equations. A particular formula like:

$$D \otimes dx^{\mu} = -\Gamma^{\mu}_{\sigma\nu}dx^{\nu} \otimes dx^{\sigma}$$

is obviously ambiguous because under the constraint $x^{\mu}dx_{\mu} = 0$ in (3.11) so that the formula above is invariant under the change:

$$\Gamma^{\rho}_{\mu\nu} \to \Gamma^{\rho}_{\mu\nu} + x^{\rho}q_{\mu\nu} + x_{(\mu}f^{\rho}_{\ \nu)}$$

where $q_{\mu\nu}$ and f^{ρ}_{ν} are arbitrary (the symmetrization of $x_{(\mu}f^{\rho}_{\ \nu)}$ is to preserve the torsion-free condition).

The reason there is some freedom is because we need to additionally impose that the connection preserves the above conditions.[5] This will subsequently fix most of the additional freedom.

We then require that these constraints are preserved by the connection:

$$D\left(x^{\mu}x_{\mu}\right) = 0, \qquad D\left(x^{\mu}p_{\mu}\right) = 0, \qquad D^2\left(x^{\mu}x_{\mu}\right) = 0, \qquad D^2\left(x^{\mu}p_{\mu}\right) = 0 \tag{3.12}$$

as well as equations coming from higher-order derivatives.

The way we will proceed is first compute the connection and curvature using (D.2) and (3.9), the ambient connection ∂ and the formula for the metric in (3.1). We then fix the additional freedom by imposing the constraints in (3.12). We will be left with a little additional freedom which will not affect any of our formulas so we make an arbitrary choice here. The result will give us the formulas in (3.15).

The conditions that Γ must satisfy are:

1. Torsion-free:
$$dx^{\sigma}\nabla_{\sigma}\left(dx^{\mu}\right) = -\Gamma^{\mu}_{\nu\sigma}dx^{\sigma}dx^{\nu} \Longrightarrow \Gamma^{\mu}_{[\nu\sigma]} = 0.$$
2. Metric-preserving:
$$\nabla_{\rho}\left(g_{\mu\nu}dx^{\mu}dx^{\nu}\right) = 0.$$
3. The directional derivative $\mathcal{D}_v$ of a vector and covector in any direction v^a is also a vector and covector respectively.
$$w_{\mu} \text{ is a covector} \iff \mathcal{D}_v w_{\mu} = v^{\rho}\left(\partial_{\rho}w_{\mu} - \Gamma^{\nu}_{\mu\rho}w_{\nu}\right) \text{ is a covector}$$
$$w^{\mu} \text{ is a vector} \iff \mathcal{D}_v w^{\mu} = v^{\rho}\left(\partial_{\rho}w^{\mu} + \Gamma^{\mu}_{\nu\rho}w^{\nu}\right) \text{ is a vector.}$$
4. The constraints in (3.10) and (3.11):
$$x^{\mu}x_{\mu} = 1/C, \qquad x^{\mu}p_{\mu} = 0 \tag{3.13}$$
$$x_{\mu}dx^{\mu} = 0, \qquad dx^{\mu}p_{\mu} + x^{\mu}dp_{\mu} = 0$$
$$\nabla_{\nu}\left(x^{\mu}dx_{\mu}\right) = 0, \qquad \nabla_{\nu}\left(p_{\mu}dx^{\mu} + x^{\mu}dp_{\mu}\right) = 0. \tag{3.14}$$

[5] To be technically correct, the constraints (3.11) come from the connection's action on the constraints (3.10).

The configuration space metric, the Christoffel symbol and the Riemann tensor for our specific case $M_{C_{p,q}}$ (and for our choice of coordinates), are using the above strategy:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} - C x_\mu x_\nu \\ \Gamma^\mu_{\nu\sigma} &= C x^\mu g_{\nu\sigma} - 2C x_{(\nu}\left(\delta^\mu_{\sigma)} - C x_{\sigma)} x^\mu\right) \\ R^\mu_{\nu\sigma\rho} &= -C\left(\delta^\mu_{[\sigma} - C x_{[\sigma} x^\mu\right) g_{\rho]\nu} \\ \omega &= \left(\delta^\mu_\nu - C x^\mu x_\nu\right)\alpha_\mu dx^\nu. \end{aligned} \tag{3.15}$$

On a general technical note, we will proceed in an identical fashion for most of the paper: and each step verify that all relevant constraints are satisfied. Although we choose a set of coordinates, even ones with constraints x^μ, the objects we consider such as ∇, g, etc. are intrinsic and coordinate independent things.

3.3. The Weyl–Heisenberg bundle

In step 2 of the algorithm, we introduce some machinery namely the operators $\hat{y}$'s to calculate the observables on general manifold M. However, unlike Fedosov who defines these $\hat{y}$'s as covectors equipped with a Moyal-like product between them we let these $\hat{y}$'s to be infinite-dimensional matrix-valued operators acting on a Hilbert space. The relations defining the $\hat{y}$'s (relations (2.6) and (2.7) in step 2) are identical in both cases.

The link to familiar Heisenberg algebras using Darboux coordinates:

The first relation (2.6) in step 2 is $\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB}$ and can be expressed in a more familiar form by a suitable choice of coordinates. In symplectic geometry there is a famous theorem, called Darboux's theorem, which states that in the neighborhood of each point on an n-dimensional symplectic manifold, there exist coordinates called Darboux coordinates $\tilde{q} = \left(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{p}_1, \ldots, \tilde{p}_n\right)$[6] where the ω takes the form:

$$\omega = d\tilde{p}_1 d\tilde{x}^1 + \cdots + d\tilde{p}_n d\tilde{x}^n.$$

In this coordinate system at $\tilde{q}$ the $\hat{y}$'s are expressed as $2n$ operators $\left(\tilde{s}^1, \ldots, \tilde{s}^n, \tilde{k}_1, \ldots, \tilde{k}_n\right)$ which have the commutators $\left[\tilde{s}^i, \tilde{s}^j\right] = \left[\tilde{k}_i, \tilde{k}_j\right] = 0$, $\left[\tilde{s}^i, \tilde{k}_j\right] = i\hbar\delta^i_j$ where i and j run from 1 through $2n$. And so at each point the $\hat{y}$'s establish a standard Heisenberg algebra (acting on a Hilbert space) which all physicists know. Therefore at each point we have a standard algebra of observables that we are intimately familiar within ordinary quantum mechanics. The full bundle of all of these algebras at all points creates a huge algebra and it is the goal of the Fedosov algorithm to choose an appropriate subalgebra in this huge algebra that we can identify as our algebra of observables subject, of course, to agreement to real physical situations. This subalgebra is the image of the map σ^{-1} on the set of all phase–space functions.

Defining properties of $\hat{y}$:

$$\begin{aligned} &\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB} \\ &D\hat{y}^A = -\Gamma^A_B \hat{y}^B = -\Gamma^A_{BC}\Theta^C \hat{y}^B, \qquad \Theta^B = \left(\theta^\sigma, \alpha_\sigma\right). \end{aligned}$$

The $\hat{y}$'s commute with the set of quantities $\{x, p, \Theta, g, \omega, \hbar, i\}$ (i.e., they behave as scalars on the matrix indices) where i is the complex unit.

*Note that the action of the phase–space connection on $\hat{y}$ is the same as the one on Θ ($D \otimes \Theta^A = \Gamma^A_{BC}\Theta^C \otimes \Theta^B$) and so we regard it as a basis of operator or matrix-valued covectors. The connection's action on the $\hat{y}$'s tells us how to parallel transport the Weyl–Heisenberg algebra (the $\hat{y}$'s) at one point to the Weyl–Heisenberg algebra of every other point in a consistent way.

[6] Note that these $2n$ coordinates are different from the $2n + 2$ embedding coordinates (x^μ, p_μ).

By defining $\hat{y}^A = \left(s^\mu, k_\mu\right)$ where the s's are the first $n+1\,\hat{y}$'s and the k's are the last $n+1\,\hat{y}$'s we have the following formula for the connection D acting on them[7] which is just plugging (3.7) and (3.8) into the equation (2.7) in step 2:

$$Ds^\mu = -\Gamma^\mu_{\ \sigma\nu} dx^\nu s^\sigma \tag{3.16}$$

$$Dk_\mu := -\frac{4}{3} R^\psi_{\ (\mu\sigma)\beta} dx^\beta s^\sigma p_\psi + \Gamma^\nu_{\ \mu\sigma} dx^\sigma k_\nu$$

$$D^2 s^\mu = dx^\psi dx^\sigma R^\mu_{\ \nu\psi\sigma} s^\nu \tag{3.17}$$

$$D^2 k_\mu = \frac{4}{3} dx^\sigma \left(C^\psi_{\ \mu\beta\nu\sigma} p_\psi dx^\nu + R^\nu_{\ (\mu\beta)\sigma} \alpha_\nu \right) s^\beta - R^\nu_{\ \mu\sigma\beta} dx^\sigma dx^\beta k_\nu$$

where again $C^c_{\ abes} := \nabla_s R^c_{\ (ab)e}$.

Introducing terminology:

In this paper when we say f is a function/form we define it to be a complex Taylor series in its variables.[8] Explicitly:

$$f(u, \ldots, v) = \sum_{l,j} f_{j_1 \cdots j_l} u^{j_1} \cdots v^{j_l} \,(j's \text{ are powers not indices})$$

where u and v are arbitrary.

So if f is a function/form of some subset or all of the quantities $x, p, \mathrm{d}x, dp, \omega, \hbar$ and i it then commutes with the $\hat{y}$'s and will be called a complex-valued function/form. On the contrary a matrix-valued function/form is a complex Taylor series in $\hat{y}$ and possibly some subset or all of the quantities $x, p, \mathrm{d}x, dp, \omega, \hbar$ and i.

So if $f(x, p, \mathrm{d}x, dp, \omega, \hbar, i)$ is a complex-valued function/form it then commutes with the $\hat{y}$'s. More explicitly with the matrix indices written (which are exceptions to our index conventions):

$$\left(\hat{y}^A \hat{y}^B\right)_{jk} = \sum_l \hat{y}^A_{jl} \hat{y}^B_{lk}$$

$$\left(\left[\hat{y}^A, f\right]\right)_{jk} := \hat{y}^A_{jk} f - f \hat{y}^A_{jk} = 0.$$

On the contrary a matrix-valued function/form does not. From now on we will not write the matrix indices explicitly.

End goal:

The idea for Fedosov's introduction of the $\hat{y}$'s is to associate to each $f(x, p) \in C^\infty(T^*M)$ a *unique* observable $\hat{f}\left(x, p, \hat{y}\right)$:

$$\hat{f}\left(x, p, \hat{y}\right) = \sum_{j,l} f_{j,l,A_1 \cdots A_l} \hbar^j \, \hat{y}^{A_1} \cdots \hat{y}^{A_l}.$$

Important note: Most of the rest of the sections will be dedicated to finding an $\hat{f}$ (i.e., the coefficients $f_{j,l,A_1\cdots A_l}$) for each $f(x, p) \in C^\infty(T^*M)$.

3.3.1. *The constant curvature case explicitly*

Specifically for $T^*M_{C_{p,q}}$ we have the induced symplectic form ω of $T^*\mathbb{R}^{n+1}$ onto $T^*M_{C_{p,q}}$ being:

$$\omega = \alpha_\mu dx^\mu = \left(\delta^\nu_\mu - C x_\mu x^\nu\right) \alpha_\mu dx^\nu.$$

From the definition of $\hat{y}$ the commutation relations in (2.6) and from the formula (3.15):

$$\left[s^\mu, s^\nu\right] = 0 = \left[k_\mu, k_\nu\right], \quad \left[s^\mu, k_\nu\right] = i\hbar \left(\delta^\mu_\nu - C x^\mu x_\nu\right).$$

[7] Note that the indices go from 1 to $2n+2$ and are different from the $2n$ operators defined above by $(\tilde{s}^1, \ldots, \tilde{s}^n, \tilde{k}_1, \ldots, \tilde{k}_n)$. The difference between them is the same as the difference between the embedding coordinates $(x^1, \ldots, x^{n+1}, p_1, \ldots, p_{n+1})$ and $(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{p}_1, \ldots, \tilde{p}_n)$.

[8] The set of all of these type of functions is sometimes called the enveloping algebra of its arguments.

Since dx^μ and α_μ are perpendicular to x the matrix counterparts s^μ and k_μ are also:

$$\eta_{\mu\nu}x^\mu s^\nu = x^\mu k_\mu = 0. \tag{3.18}$$

Since $\eta_{\mu\nu}x^\mu s^\nu = x^\mu k_\mu = 0$ we have n independent operators which is required (one for each direction on $M_{C_{p,q}}$).

The action of the connection and curvature acting on s^μ & k_μ on a general phase–space and not just $T^*M_{C_{p,q}}$ is written down directly using the formulas in (3.15) into the formula (3.16).

3.4. Constructing the global derivation

In step 3 in the algorithm, must determine a global derivation as a matrix commutator $\hat{D} = \left[\hat{Q}, \cdot\right]$ which is central to constructing the coefficients $f_{A_1\cdots A_l}$ in equation (2.10) for each $f(x, p) \in C^\infty(T^*M)$.

Define the derivation $\hat{D}$ by the graded commutator:

$$\hat{D} = \left[\hat{Q}, \cdot\right]/i\hbar = \left[\hat{Q}_A \Theta^A, \cdot\right]/i\hbar$$
$$\hat{Q}_A = \sum_l Q_{AA_1\cdots A_l}\hat{y}^{A_1}\cdots\hat{y}^{A_l}$$

where $\Theta^B = (dx^\sigma, \alpha_\sigma)$ (see again the definition for α in (3.6)) and $Q_{AA_1\cdots A_l}$ are complex-valued functions of x and p that need to be determined. We reiterate that complex-valued functions are not matrices hence they commute with the $\hat{y}$'s.

In step 3 we have the mysterious condition (2.8) that partially determines the functions $Q_{AA_1\cdots A_l}$:

We rewrite the condition (2.8) as:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = \left[\Omega - D\hat{Q} + \hat{Q}^2/i\hbar, \hat{y}^A\right]/i\hbar = 0 \tag{3.19}$$

where Ω is the phase–space curvature as a commutator (see [2] for the details):

$$\frac{1}{i\hbar}\left[\Omega, \hat{y}^A\right] := D^2\hat{y}^A = R_B{}^A\hat{y}^B \tag{3.20}$$

with solution $\Omega := -\frac{1}{2}\omega_{AC}R_{CE}{}^A_B\Theta^C \wedge \Theta^E\hat{y}^B\hat{y}^C$ where $R_{CE}{}^A_B$ is the phase–space curvature.

From now on we let[9]:

$$\Omega - D\hat{Q} + \hat{Q}^2/i\hbar = 0 \tag{3.21}$$

and keep it in the back of our minds that we could add something that commutes with all $\hat{y}$'s to $\Omega - D\hat{Q} + \hat{Q}^2/i\hbar$.

To emphasize the importance of this equation the reader should note that the whole Fedosov $*$ hinges on this $\hat{Q}$ existing. We know that a solution exists perturbatively in general (the recursive solution for it is in [2] on p. 144), however, convergence issues of the general series still remain. We have found that solving for $\hat{Q}$ to be the hardest point of the computation of the Fedosov $*$ because of the need for the right ansatz to the nonlinear equation (3.21).

Fedosov at this point would implement an algorithm to construct $\hat{Q}$ perturbatively, however, rather than do this we will make an ansatz for $\hat{Q}$ using some ingenuity. This will give us an exact solution for $\hat{Q}$.

Ω is:

$$\begin{aligned}\Omega &= -R^\nu{}_{\mu\sigma\beta}dx^\sigma dx^\beta k_\nu s^\mu + \frac{2}{3}D\left(R^\nu{}_{(\mu\beta)\sigma}p_\nu s^\beta s^\mu dx^\sigma\right)\\ &= -R^\nu{}_{\mu\sigma\beta}dx^\sigma dx^\beta k_\nu s^\mu + \frac{2}{3}dx^\sigma\left(C^\psi{}_{\mu\beta\nu\sigma}p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma}\alpha_\nu\right)s^\beta s^\mu\end{aligned} \tag{3.22}$$

where $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$.

[9] This is the same as the condition of Fedosov $\Omega - Dr + \hat{d}r + r^2 = 0$. See [2,8].

We verify that it gives the curvature as commutators:

$$\frac{1}{i\hbar}\left[\Omega, s^{\mu}\right] = D^2 s^{\mu} = R^{\mu}{}_{\nu\psi\varepsilon} dx^{\psi} dx^{\varepsilon} s^{\nu}$$
$$\frac{1}{i\hbar}\left[\Omega, k_{\mu}\right] = D^2 k_{\mu} = \frac{4}{3} dx^{\sigma}\left(C^{\psi}{}_{\mu\beta\nu\sigma} p_{\psi} dx^{\nu} + R^{\nu}{}_{(\mu\beta)\sigma}\alpha_{\nu} dx^{\beta}\right) s^{\beta} - R^{\nu}{}_{\mu\sigma\beta} dx^{\sigma} dx^{\beta} k_{\nu}.$$

Our ansatz for a solution to equation (3.21) is:

$$\hat{Q} = \left(s^{\mu}\alpha_{\mu} - z_{\mu} dx^{\mu}\right) + j^{\mu}\alpha_{\mu} + z_{\nu} f^{\nu}_{\mu} dx^{\mu} + p_{\nu}\left(\left(D + f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma} - dx^{\sigma}\hat{\partial}_{\sigma}\right) j^{\nu} + \Gamma^{\nu}_{\rho\sigma} dx^{\sigma} j^{\rho} - \frac{2}{3} R^{\nu}{}_{(\mu\beta)\sigma} s^{\beta} s^{\mu} dx^{\sigma}\right) \tag{3.23}$$

where $\hat{\partial}_{\mu} := \partial/\partial s^{\mu}$ and along with the condition on f^{ν}_{μ}:

$$\left(\left(D + f^{\mu}_{\rho} dx^{\rho}\hat{\partial}_{\mu} - dx^{\mu}\hat{\partial}_{\mu}\right) f^{\nu}_{\sigma} + \Gamma^{\nu}_{\rho\mu} dx^{\mu} f^{\rho}_{\sigma} - \Gamma^{\nu}_{\sigma\mu} dx^{\mu} + R^{\nu}{}_{\mu\beta\sigma} s^{\mu} dx^{\beta}\right) dx^{\sigma} = 0. \tag{3.24}$$

To see that the term:

$$p_{\nu}\left(\left(D + f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma} - \hat{\partial}_{c}\right) j^{\nu} + \Gamma^{\nu}_{\rho\sigma} dx^{\sigma} j^{\rho}\right)$$

in (3.23) is coordinate independent if j^a and f^e_c are, we express it in terms of abstract indices:

$$p_b\left(\nabla_c j^b + f^e_c\hat{\partial}_e - \hat{\partial}_c\right) j^b$$

where we used the fact that $Dj^b = \Theta^C D_C j^b = \nabla_c j^b$ because j^b is a function of x and s only. So we can see that if j^a and f^e_c are independent of the choice of configuration space coordinates then so is $\hat{Q}$.

By putting (3.23) and (3.24) into equation (3.21) and performing a straightforward calculation we can easily verify that they solve the equation in (3.21). Moreover, equation (3.24) is locally integrable for f^{ν}_{μ} by the Cauchy–Kovalevskaya theorem (see Appendix B). This fact allows us to come up with an iterative solution in the spirit of the series of Fedosov star-product.

We have therefore proved the following theorem:

Theorem 1. *Given any cotangent bundle T^*M, the solution to the equation in (3.21) is (3.23) along with the condition in (3.24) where the equation (3.24) is locally integrable for f^{ν}_{μ} by the Cauchy–Kovalevskaya theorem.*

3.4.1. The constant curvature case explicitly

The solution that was found for our example of $T^*M_{C_{p,q}}$ using the above ansatz (3.23) and condition (3.24):

$$\hat{Q} = \left(s^{\mu}\alpha_{\mu} - z_{\mu} dx^{\mu}\right) - C\left(z_{\nu} s^{\nu}\right)\left(s_{\mu} dx^{\mu}\right) + \frac{C}{3}\left(\left(p_{\nu} s^{\nu}\right)\left(s_{\mu} dx^{\mu}\right) - \left(p_{\nu} dx^{\nu}\right) u\right) \tag{3.25}$$

where $z_{\mu} := k_{\mu} + p_{\mu}$, $u = \eta_{\mu\nu} s^{\mu} s^{\nu}$ and $p_{\mu} x^{\mu} = \eta_{\mu\nu} s^{\mu} x^{\nu} = k_{\mu} x^{\mu} = \alpha_{\mu} x^{\mu} = \eta_{\mu\nu} dx^{\mu} x^{\nu} = 0$.

3.5. The basis for the algebra of observables

Now we have all the tools in place to associate an observable $\hat{f}$ to every $f \in C^{\infty}(T^*M)$. At step 4 in our algorithm we require that every observable $\hat{f}\left(x, p, \hat{y}\right)$ must satisfy Eq. (2.10).

The condition $f_{j,l,(A_1\cdots A_l)} = f_{j,l,A_1\cdots A_l}$ is the condition for Weyl or symmetric quantization. You can choose another ordering, but this is sufficient. Moreover, this is the choice that Fedosov makes for ordering. The condition in (2.10) is used to solve for a unique $\hat{f}$ to every $f \in C^{\infty}(T^*M)$ up to some "reasonable" ambiguity.

Here we (again) diverge from the Fedosov algorithm. Instead of constructing the coefficients so that $f_{j,l,A_1\cdots A_l}$ is symmetric in $\hat{y}$'s we instead require that each term in:

$$\hat{f}\left(\hat{x}, \hat{p}\right) = \sum_{jlm} \tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}\,\hbar^j\,\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\,\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m} \tag{3.26}$$

where $\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}$ is a complex-valued function of x and p and is symmetric in all $\hat{x}$ and $\hat{p}$, i.e.:

$$\hat{f}\left(\hat{x},\hat{p}\right)=\sum_{lm}\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}\,\hbar^j\,SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\right)$$

where:

$$\begin{aligned}SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\right)&=\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}+\left(\text{all perms. of }\hat{x}'s\text{ and }\hat{p}'s\right)\\&=\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}+\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_{l-1}}\hat{p}_{\nu_1}\hat{x}^{\mu_l}\hat{p}_{\nu_2}\cdots\hat{p}_{\nu_m}+\cdots.\end{aligned}$$

The definition of $\hat{f}$ in (3.26) corresponds to the phase–space function:

$$\sigma\left(\hat{f}\right)=f\left(x,p\right)=\sum_{jlm}\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}\,\hbar^j\,x^{\mu_1}\cdots x^{\mu_l}p_{\nu_1}\cdots p_{\nu_m}.$$

The nice property of the above form of (3.26) is that the coefficients $\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}$ are constant. This is easily seen by acting $(D-\hat{D})$ on the equation. Also, the formula is nice because now we can find any basis $\left(\hat{x},\hat{p}\right)$ and these will give unique $\hat{f}$ for all phase–space function f. All we need to do now is to find any basis $\left(\hat{x},\hat{p}\right)$ which is our next task.

Finding a basis:

We define a basis $\left(\hat{x},\hat{p}\right)$ as any operator of the form:

$$\hat{x}^{\mu}=\sum_{l}b^{\mu}_{j,l,A_1\cdots A_l}\,\hbar^j\,\hat{y}^{A_1}\cdots\hat{y}^{A_l}\tag{3.27}$$

$$\hat{p}_{\mu}=\sum_{l}c_{j,l,\mu,A_1\cdots A_l}\,\hbar^j\,\hat{y}^{A_1}\cdots\hat{y}^{A_l}\tag{3.28}$$

where $b^{\mu}_{j,l,A_1\cdots A_l}$ and $c_{j,l,\mu,A_1\cdots A_l}$ are complex-valued functions of x and p (which are the coefficients $f_{j,l,A_1\cdots A_l}$ in Eq. (2.10) where $f=x$ or $f=p$ respectively) and will be partially determined by the equations:

$$\left(D-\hat{D}\right)\hat{x}^{\mu}=0,\quad \sigma\left(\hat{x}^{\mu}\right)=b^{\mu}_{0,0}=x^{\mu}\tag{3.29}$$

$$\left(D-\hat{D}\right)\hat{p}_{\mu}=0,\qquad \sigma\left(\hat{p}_{\mu}\right)=c_{0,0,\mu}=p_{\mu}.\tag{3.30}$$

Remember that our observables are defined in (2.9). To express them in the form of (3.26) we need to invert the relations (3.29) and (3.30) so that we express $\hat{y}$ in terms of x, p, $\hat{x}$, and $\hat{p}$ as a the matrix-valued function $\hat{y}^A=\hat{y}^A\left(x,p,\hat{x},\hat{p}\right)$. By substituting $\hat{y}^A=\hat{y}^A\left(x,p,\hat{x},\hat{p}\right)$ into (2.9) it will be observed that all observables can be expressed in the form of (3.26). Of course, the caveat is that we have assumed the convergence of all of these series which will not be true in general.

To construct a basis $\left(\hat{x},\hat{p}\right)$ for the algebra Fedosov at this point would implement an algorithm yielding perturbative solutions (see [2] p 146). We instead try to find exact solutions to them.[10]

3.5.1. The constant curvature case explicitly

Specifically for the case of $T^*M_{C_{p,q}}$ we make the ansatz for both $\hat{x}$ and $\hat{p}$:

$$\begin{aligned}\hat{x}^{\mu}&=f\left(u\right)x^{\mu}+h\left(u\right)s^{\mu}\\\hat{p}_{\mu}&=z_{\nu}s^{\nu}x_{\mu}g\left(u\right)+z_{\mu}j\left(u\right)\end{aligned}$$

where $u:=\eta_{\mu\nu}s^{\mu}s^{\nu}$ and $z_{\mu}:=k_{\mu}+p_{\mu}$.

[10] We, again, ran the Fedosov algorithm a few times to help us see what the ansatz should take.

We require that both $\hat{x}$ and $\hat{p}$ satisfy the two conditions (3.29) and (3.30) and by solving the subsequent differential equations we obtain the solutions:

$$\hat{x}^{\mu} = \left(x^{\mu} + s^{\mu}\right) \frac{1}{\sqrt{Cu+1}} \tag{3.31}$$

$$\hat{p}_{\mu} = \left(-C z_{\nu} s^{\nu} x_{\mu} + z_{\mu}\right) \sqrt{Cu+1} - iC\hbar n \hat{x}_{\mu} \tag{3.32}$$

where $u = s_{\mu} s^{\mu}$, $z_{\mu} := k_{\mu} + p_{\mu}$, and with the computed conditions:

$$\begin{aligned} &\sigma\left(\hat{x}^{\mu}\right) = b^{\mu}_{0,0} = x^{\mu}\ , \qquad \sigma\left(\hat{p}_{\mu}\right) = c_{0,0,\mu} = p_{\mu} \\ &\hat{x}\cdot\hat{x} = 1/C, \qquad \hat{x}\cdot\hat{p} = \hat{p}\cdot\hat{x} - ni\hbar = 0. \end{aligned} \tag{3.33}$$

We now use these results to write the solution for $\hat{x}$ and $\widehat{\tilde{p}}$ for the embedding:

$$x^{\mu} x_{\mu} = 1/C, \qquad x^{\mu} \tilde{p}_{\mu} = A.$$

Since this is a canonical transformation:

$$\tilde{p}_{\mu} = p_{\mu} + CAx_{\mu}, \qquad \tilde{x}^{\mu} = x^{\mu}$$

and preserves all constraints except $x^{\mu} p_{\mu} = A$ we can write the solution as:

$$\widehat{\tilde{p}}_{\mu} = \hat{p}_{\mu} + CA\hat{x}_{\mu}, \qquad \widehat{\tilde{x}}^{\mu} = \hat{x}^{\mu}$$

(see Appendix C for proof) so:

$$\begin{aligned} \widehat{\tilde{x}}^{\mu} &= \hat{x}^{\mu} = \left(x^{\mu} + s^{\mu}\right) \frac{1}{\sqrt{Cu+1}} \\ \widehat{\tilde{p}}_{\mu} &= \left(-C z_{\nu} s^{\nu} x_{\mu} + z_{\mu}\right) \sqrt{Cu+1} + C\left(A - i\hbar n\right) \hat{x}_{\mu}. \end{aligned}$$

Note: From now on we will use the embedding (and by dropping the tilde):

$$x^{\mu} x_{\mu} = 1/C, \quad x^{\mu} p_{\mu} = A$$

and the solutions:

$$\hat{x}^{\mu} = \left(x^{\mu} + s^{\mu}\right) \frac{1}{\sqrt{Cu+1}} \tag{3.34}$$

$$\hat{p}_{\mu} = \left(-C z_{\nu} s^{\nu} x_{\mu} + z_{\mu}\right) \sqrt{Cu+1} + C\left(A - i\hbar n\right) \hat{x}_{\mu} \tag{3.35}$$

with computed conditions:

$$\hat{x}\cdot\hat{x} = 1/C, \qquad \hat{x}\cdot\hat{p} = \hat{p}\cdot\hat{x} - ni\hbar = A. \tag{3.36}$$

In group-theoretic terminology the two conditions above represent the Casimir invariants of the algebra of observables.

3.6. The commutators

Once we have $\hat{x}^{\mu}$ and $\hat{p}_{\mu}$, i.e., the coefficients $b^{\mu}_{j,l,A_1\cdots A_l}$ and $c_{j,l,\mu,A_1\cdots A_l}$ we work out the commutation relations $\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right], \left[\hat{x}^{\mu}, \hat{p}_{\nu}\right]$ and $\left[\hat{p}_{\mu}, \hat{p}_{\nu}\right]$ using the solution for $\hat{x}$ and $\hat{p}$ (for either case they are (3.31) and (3.32)) in a brute force calculation:

$$\hat{h}\left(\hat{x}, \hat{p}\right) := \left[\hat{f}\left(\hat{x}, \hat{p}\right), \hat{g}\left(\hat{x}, \hat{p}\right)\right]$$

$$\Longrightarrow \left[f_{*}\left(x, p\right), g_{*}\left(x, p\right)\right]_{*} = h_{*}\left(x, p\right) = i\hbar\left[f, g\right]_{P} + O\left(\hbar^{2}\right) \tag{3.37}$$

where $\hat{f}$, $\hat{g}$, $\hat{h}$ and f_*, g_*, h_* are functions defined by:

$$\hat{f}\left(\hat{x},\hat{p}\right)=\sum_{lmj}\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}\hbar^j\,\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}$$
$$f_*\left(x,p\right)=\sum_{lmj}\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}\hbar^j\,x^{\mu_1}*\cdots*x^{\mu_l}*p_{\nu_1}*\cdots*p_{\nu_m}$$

where $\tilde{f}^{\nu_1\cdots\nu_m}_{j,l,m,\mu_1\cdots\mu_l}$ are constants.

These two sets, one of all f_*'s $\{f_*\}$ and one of all $\hat{f}$'s $\left\{\hat{f}\right\}$ defined above are isomorphic with isomorphism σ^{-1}.

3.6.1. The constant curvature case explicitly

In our case of $T^*M_{C_{p,q}}$ we compute:

$$\begin{aligned}&\left[\hat{x}^\mu,\hat{x}^\nu\right]=0\\&\left[\hat{x}^\mu,\hat{p}_\nu\right]=i\hbar\left(\delta^\mu_\nu-C\hat{x}^\mu\hat{x}_\nu\right)\\&\left[\hat{p}_\mu,\hat{p}_\nu\right]=2i\hbar C\hat{x}_{[\nu}\hat{p}_{\mu]}\end{aligned}\tag{3.38}$$

along with the computed conditions:

$$\hat{x}^\mu\hat{x}_\mu=1/C,\qquad \hat{p}_\mu\hat{x}^\mu+ni\hbar=\hat{x}^\mu\hat{p}_\mu=A.$$

We now define:

$$\hat{M}_{\mu\nu}=\hat{x}_{[\mu}\hat{p}_{\nu]}=\hat{p}_{[\nu}\hat{x}_{\mu]}=\left(-Cz_\rho s^\rho x_{[\nu}+z_{[\nu}\right)\left(x_{\mu]}+s_{\mu]}\right).$$

The leading order term is found to be:

$$\sigma\left(\hat{M}_{\mu\nu}\right)=x_{[\mu}p_{\nu]}=M_{\mu\nu}.$$

We recognize that $\hat{M}$ and $\hat{x}$ are the more "natural" variables than $\hat{x}$ and $\hat{p}$ because $\hat{p}_\mu\hat{x}^\mu=-ni\hbar$ and $\hat{x}^\mu\hat{p}_\mu=A$ where A is an arbitrary constant. These are very "unnatural" since there is no reason why it should not be $\hat{p}_\mu\hat{x}^\mu=A$ and $\hat{x}^\mu\hat{p}_\mu=ni\hbar$ or something else like this. $\hat{M}$ projects out the part of the momentum $\hat{p}$ that is parallel to $\hat{x}$ ($2\hat{x}^\mu\hat{M}_{\mu\nu}=\hat{p}_\nu/C-A\hat{x}_\nu$). We regard this part of $\hat{p}$ to be irrelevant because it does not affect the form of the commutators in (3.38) and it preserves the symplectic form.

We have the definitions:

$$\begin{aligned}&\hat{x}^\mu=\left(x^\mu+s^\mu\right)\frac{1}{\sqrt{Cu+1}}\\&\hat{M}_{\mu\nu}=\hat{x}_{[\mu}\hat{p}_{\nu]}=\hat{p}_{[\nu}\hat{x}_{\mu]}=-Cz_\rho s^\rho x_{[\nu}s_{\mu]}+z_{[\nu}x_{\mu]}+z_{[\nu}s_{\mu]}\end{aligned}\tag{3.39}$$

and the computed commutation relations (which is again very straightforward):

$$\begin{aligned}&\left[\hat{x}^\mu,\hat{x}^\nu\right]=0\\&\left[\hat{x}_\mu,\hat{M}_{\nu\rho}\right]=i\hbar\hat{x}_{[\nu}\eta_{\rho]\mu}\\&\left[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}\right]=i\hbar\left(\hat{M}_{\sigma[\mu}\eta_{\nu]\rho}-\hat{M}_{\rho[\mu}\eta_{\nu]\sigma}\right)\end{aligned}\tag{3.40}$$

subject to the conditions:

$$\hat{x}^\mu\hat{x}_\mu=1/C,\qquad \hat{M}_{\mu\nu}=-\hat{M}_{\nu\mu},\qquad 2\hat{x}^\mu\hat{M}_{\mu\nu}=\hat{p}_\nu/C-A\hat{x}_\nu.\tag{3.41}$$

We then see that the M's generate $\mathbb{SO}\,(p+1,q)$ in the case of $C>0$ because $\operatorname{sign}(\eta)=(p+1,q)$. Similarly the M's generate $\mathbb{SO}\,(p,q+1)$ in the case of $C<0$ because $\operatorname{sign}(\eta)=(p,q+1)$. We expected to see these groups in the group of observables because they are the symmetry groups for hyperboloids defined by $x^\mu x_\mu=1/C$.

The enveloping algebra of these operators gives the algebra of observables on $T^*M_{C_{p,q}}$ a general element being:

$$\hat{f}\left(\hat{x},\hat{M}\right)=\sum_{lm}\tilde{f}^{\nu_1\cdots\nu_{2m}}_{j,l,m,\mu_1\cdots\mu_l}SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{M}_{\nu_1\nu_2}\cdots\hat{M}_{\nu_{(2m-1)}\nu_{2m}}\right)$$

where the coefficients $f^{\nu_1\cdots\nu_{2m}}_{\mu_1\cdots\mu_l}$ are constants.

3.7. The algebra of observables is the enveloping algebra of a pseudo-orthogonal group

Now that we have a basis of the algebra of observables, we want to analyze the Lie group associated to the Lie algebra relations in (3.40). It turns out that the group is $\mathbb{SO}\,(p+1,q+1)$.

The commutation relations in (3.41) are computed to be equivalent to:

$$\begin{aligned}&\left[\hat{M}_{\mu'\nu'},\hat{M}_{\rho'\sigma'}\right]=i\hbar\left(\hat{M}_{\rho'[\mu'}\eta_{\nu']\sigma'}-\hat{M}_{\sigma'[\mu'}\eta_{\nu']\rho'}\right)\\&\hat{M}_{\mu'\nu'}=-\hat{M}_{\nu'\mu'}\end{aligned}\tag{3.42}$$

where we use the notation that the primed indices run from $1,\ldots,n+2$. Thus the $\hat{M}'$'s (i.e., the $\hat{M}_{\mu'\nu'}$'s) form the Lie Algebra of $\mathbb{SO}\,(p+1,q+1)$, $\mathfrak{so}\,(p+1,q+1)$ for both $C>0$ and $C<0$!

The extra $n+1$ generators of $\hat{M}$ being:

$$\hat{M}_{(n+2)\mu'}=-\hat{M}_{\mu'(n+2)}=\frac{1}{2\sqrt{|C|}}\hat{p}_{\mu'}=\frac{1}{2\sqrt{|C|}}\left(C\hat{x}^{\nu}\hat{M}_{\nu\mu'}-\frac{CA}{i\hbar}\hat{x}_{\mu'}\right)\text{ for }\mu'=1,\ldots,n+1$$
$$\hat{M}_{(n+2)(n+2)}=0$$

along with the extra components of η being:

$$\eta_{(n+2)(n+2)}=-C/|C|$$
$$\eta_{(n+2)\mu'}=0\text{ for }\mu'\neq n+2.$$

It is a straightforward computation to verify that the commutation relation $\left[\hat{M}_{\mu'\nu'},\hat{M}_{\rho'\sigma'}\right]$ is the above.

The summary of the results:

We now have the following scheme worked out exactly:

- For the configuration space $M_{C_{p,q}}$ with $\text{sign}(g)=(p,q)$ and $C>0$:

  $\Longrightarrow \text{sign}\,(\eta)=(p+1,q)$, M generate $\mathbb{SO}\,(p+1,q)$

  $\Longrightarrow \text{sign}\left(\eta'\right)=(p+1,q+1)$, $M'=(M,x)$ generate $\mathbb{SO}(p+1,q+1)$.

- For the configuration space $M_{C_{p,q}}$ with $\text{sign}(g)=(p,q)$ and $C<0$:

  $\Longrightarrow \text{sign}\,(\eta)=(p,q+1)$, M generate $\mathbb{SO}\,(p,q+1)$

  $\Longrightarrow \text{sign}\left(\eta'\right)=(p+1,q+1)$, $M'=(M,x)$ generate $\mathbb{SO}\,(p+1,q+1)$.

3.8. A summary of results for de sitter and Anti-de sitter space–times

Here we give a summary of the results we have obtained for the de Sitter and Anti-de Sitter (dS/AdS) space–times. In the next subsection we will state the more general results obtained in this paper which is a straightforward generalization of this case.

We first embed dS/AdS in a flat five-dimensional space given by the embedding formulas:

$$\eta_{\mu\nu}x^{\mu}x^{\nu}=1/C\quad\text{and}\quad x^{\mu}p_{\mu}=A$$

where C and A are some real arbitrary constants, and η is the embedding flat metric. For dS $\eta=\text{diag}\,(1,-1,-1,-1,-1)$, $C<0$ and AdS $\eta=\text{diag}\,(1,1,-1,-1,-1)$, $C>0$.

We obtained the exact results for the Fedosov star-commutators:

$$\begin{aligned}&\left[x^{\mu},x^{\nu}\right]_{*}=0\quad [x_{\mu},M_{\nu\rho}]_{*}=i\hbar x_{[\nu}\eta_{\rho]\mu}\\&[M_{\mu\nu},M_{\rho\sigma}]_{*}=i\hbar(M_{\rho[\mu}\eta_{\nu]\sigma}-M_{\sigma[\mu}\eta_{\nu]\rho})\end{aligned}\tag{3.43}$$

indices run from 0 to 4, $M_{\mu\nu}=x_{[\mu}*p_{\nu]}$, $x_{\mu}=\eta_{\mu\nu}x^{\nu}$.

The conditions of the embedding $x^{\mu}x_{\mu},\ x^{\mu}p_{\mu}$ become the Casimir invariants of the algebra in group theoretic language.

We now summarize our two key observations:

1. M's generate $\mathbb{SO}(1,4)$ and $\mathbb{SO}(2,3)$ for dS and AdS respectively.
2. M's and x's generate $\mathbb{SO}(2,4)$ for *both* dS and AdS.

By calculating $R=-16C$ and $p_{\mu}*p^{\mu}$ in terms of M and x the Hamiltonian (2.5) is:

$$H=2CM_{\mu\nu}*M^{\mu\nu}+(A-4i\hbar)\,AC-16\xi C\tag{3.44}$$

where $M_{\mu\nu}*M^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(1,4)$ or $\mathbb{SO}(2,3)$ for dS or AdS respectively.

In the more familiar form of Hilbert space language the KG equation (2.3) takes the form:

$$(2C\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}+\chi C)|\phi_m\rangle=m^2|\phi_m\rangle\tag{3.45}$$

where $\langle\phi_m|\phi_m\rangle=1$, $\mathbb{C}\ni\chi=(A-4i\hbar)\,A-16\xi$ is an arbitrary constant, and we regard all groups to be in a standard irreducible representation on the set of linear Hilbert space operators.

These subgroups are the symmetry groups of the manifolds for dS or AdS respectively. Again, $\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(1,4)$ or $\mathbb{SO}(2,3)$ for dS or AdS respectively. Therefore, the above KG equation (3.45) states that the eigenstates of mass $|\phi_m\rangle$ label the different representations of $\mathbb{SO}(1,4)$ and $\mathbb{SO}(2,3)$ for dS and AdS respectively sitting inside the full group of observables $\mathbb{SO}(2,4)$ which is confirmed by the well-known results of Frønsdal [3–6] as well as others.

E.g. In the case of spin 0 particles the operator $\hat{M}^2$ becomes the Laplace–Beltrami operator $\nabla_{\mu}\nabla^{\mu}$ and $\hat{x}^{\mu}\hat{p}_{\mu}\to -i\hbar x^{\mu}\nabla_{\mu}$ so let $\phi(x):=\langle x|\phi\rangle$ then:

$$\left(2i\hbar C\nabla_{\mu}\nabla^{\mu}-\chi C-m^2\right)\phi\,(x)=0$$

where $-i\hbar x^{\mu}\nabla_{\mu}\phi=A\phi$. This equation is the free wave equation on AdS that is studied in [4] and therefore the results given here are consistent with what has been done previously.

3.9. The algebra of observables and the Klein–Gordon (KG) equation in our case

This subsection is a straightforward generalization of the last subsection. This summarizes the main results of this paper in its most general form.

We rewrite $p_{\mu}*p^{\mu}$ in terms of the generators of all groups and subgroups (i.e., x's and the M's) and the Casimir invariants of the these groups and subgroups.

It is well-known that the Casimir invariants of the subgroup generated by M are:

$$\begin{aligned}M^2&:=M_{\mu\nu}*M^{\mu\nu}\\M^4&:=M_{\mu_1\mu_2}*M^{\mu_2\mu_3}*M_{\mu_3\mu_4}*M^{\mu_4\mu_1}\\&\ \vdots\\M^N&:=M_{\mu_1\mu_2}*M^{\mu_2\mu_3}*\cdots*M_{\mu_{N-1}\mu_N}*M^{\mu_N\mu_1}\end{aligned}$$

where N is the integer part of $\frac{p+q+1}{2}$, i.e., the rank of the group $\mathbb{SO}(p+1,q)$ or $\mathbb{SO}(p,q+1)$.

Also, the Casimir invariants of the full group $\mathbb{SO}(p+1, q+1)$ are:

$$\begin{aligned}
M'^2 &:= M_{\mu'\nu'} * M^{\mu'\nu'} \\
M'^4 &:= M_{\mu'_1\mu'_2} * M^{\mu'_2\mu'_3} * M_{\mu'_3\mu'_4} * M^{\mu'_4\mu'_1} \\
&\ \ \vdots \\
M'^{N'} &:= M_{\mu'_1\mu'_2} * M^{\mu'_2\mu'_3} * \cdots * M_{\mu'_{N'-1}\mu'_{N'}} * M^{\mu'_{N'}\mu'_1}
\end{aligned}$$

where N' is the integer part of $\frac{p+q+2}{2}$, i.e., the rank of the group $\mathbb{SO}(p+1, q+1)$.

Using the equation $M_{\mu\nu} = x_{[\mu} * p_{\nu]}$ we compute directly:

$$\begin{aligned}
M'^2 &= -\frac{1}{2}(A - i\hbar n) A \\
M^2 &= \frac{1}{2C} p_\mu * p^\mu + M'^2 \Longrightarrow p_\mu * p^\mu = 2C\left(M^2 - M'^2\right)
\end{aligned}$$

using $\left[x^\mu, p_\mu\right]_* = i\hbar\left(\delta^\mu_\mu - 1\right) = n$ is the dimension of M, $R = -n^2 C$, $x^\mu * x_\mu = 1/C$, and $x^\mu * p_\mu = A$.

So by calculating $R = -n^2 C$ and $p_\mu * p^\mu$ in terms of M and x the Hamiltonian (2.5) is:

$$H = p_\mu * p^\mu + \xi R = 2CM^2 + C(A - i\hbar n) A - n^2 \xi C \tag{3.46}$$

where $M_{\mu\nu} * M^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(p, q+1)$ or $\mathbb{SO}(p+1, q)$ for $T^* M^-_{C_{p,q}}$ and $T^* M^+_{C_{p,q}}$ respectively. In addition these subgroups are the symmetry groups of the manifolds for $T^* M^-_{C_{p,q}}$ and $T^* M^+_{C_{p,q}}$ respectively.

Using the correspondence between a Hilbert space formulation and DQ given by Fedosov as mentioned in the last section we reformulate (2.4) into the form of (2.3).

In the more familiar form of Hilbert space language the KG equation (2.3) takes the form:

$$(2C\hat{M}_{\mu\nu}\hat{M}^{\mu\nu} + \chi C)|\phi_m\rangle = m^2|\phi_m\rangle \tag{3.47}$$

where $\langle\phi_m|\phi_m\rangle = 1$, $\mathbb{C} \ni \chi = (A - i\hbar n) A - n^2\xi$ is an arbitrary constant, and we regard all groups to be in a standard irreducible representation on the set of linear Hilbert space operators. Again, $\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(p, q+1)$ or $\mathbb{SO}(p+1, q)$ for $T^* M^-_{C_{p,q}}$ or $T^* M^+_{C_{p,q}}$ respectively.

Therefore, the above KG equation (3.47) states that the eigenstates of mass $|\phi_m\rangle$ label the different representations of $\mathbb{SO}(p, q+1)$ and $\mathbb{SO}(p+1, q)$ for dS and AdS respectively sitting inside the full group of observables $\mathbb{SO}(p+1, q+1)$.

E.g. In the case of spin 0 particles in n-dimensions the operator $\hat{M}^2$ becomes the Laplace–Beltrami operator $\nabla_\mu\nabla^\mu = (-g)^{1/2}\,\partial_\mu g^{\mu\nu}(-g)^{-1/2}\,\partial_\nu$ and $\hat{x}^\mu\hat{p}_\mu \to -i\hbar x^\mu\nabla_\mu$, so let $\phi(x) := \langle x|\phi\rangle$ then:

$$\left(2C\nabla_\mu\nabla^\mu - C\chi - m^2\right)\phi(x) = 0$$

where $x^\mu\nabla_\mu\phi = A\phi$. This equation is the free scalar wave equation on $T^* M^\pm_{C_{p,q}}$.

4. Conclusions

In conclusion, the results of this paper confirm the well-known results for the Klein–Gordon equation in [3–6] as well as many others. The difference is that we confirmed these results in the context of DQ. The beautiful thing about these computations is that they are algorithmic and they can be done for any manifold, whereas some previous techniques in quantization relied heavily on the symmetries of these particular manifolds or the type of dynamical evolutions studied. We note that while we expected the symmetry group of the observables $\mathbb{SO}(q, p+1)$ or $\mathbb{SO}(q+1, p)$ to be in this group we did not expect that the full group of observables to be $\mathbb{SO}(q+1, p+1)$. This fact may be well-known to group theorists, however it was surprising to us. In the dS/AdS this is the group $\mathbb{SO}(2, 4)$ this we suspect is the conformal group of the manifold $\mathbb{SO}(2, 4)$ but a clear interpretation is needed to assert this claim.

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Appendix A. Notations

Here I will briefly list my definitions and notations:

(1) **Index notations:**
  (a) We use the convention that the lower-case indices run from $1, \ldots, n$ (space–time indices) and capital ones run from $1, \ldots 2n$ (phase–space indices).
  (b) We employ the abstract index notation for this paper for lower-case indices only. Lower-case Greek letters are numerical indices while lower-case Latin letters are abstract ones. (See [13].)
  (c) The abstract indices that are not written will form indices so that multiplication of them implies a wedging $\wedge$ of the forms.
  Abstract indices convention:
  When we write $D = \Theta^B D_B$ and this acts on some configuration space quantity like a one-form $v_a = v_\mu(x)dx^\mu$ (on the configuration space) in the operator D the tensor index is suppressed. We therefore make the convention that in the abstract index notation the label B in $D = \Theta^B D_B$ will determine the abstract index of the configuration space quantity as b. For example:

$$D \otimes v_a = \Theta^B D_B \otimes v_a = \nabla_b v_a$$

  (d) Some exceptions to our index convention is needed. The letters j, l, m, k will **always** be reserved for labelling powers and other numerical labelling including non-space–time indices and thus will not go according to our index conventions in (a) and (b).

(2) **Raising and lowering indices:** We will always raise and lower the lower-case indices or $M_{C_{p,q}}$ indices (Greek or Latin) by the metric of the imbedding space $\eta_{\mu\nu}$. We will always raise and lower the upper-case indices with the symplectic form ω_{AB}.

(3) **Constant curvature manifold of codimension one:**
An n-dimensional constant curvature manifold embedded in a $(n+1)$-dimensional flat space $(\mathbb{R}^{n+1})$ given by an embedding

$$\eta_{\mu\nu}x^\mu x^\nu = 1/C, \qquad x^\mu p_\mu = A$$

where $\mu = 1, \ldots, n+1$.

$$T^*M^+_{C_{p,q}} = T^*M_{C_{p,q}} \text{ with } C > 0\ , \qquad \text{sign}(g) = (p,q)\,, \qquad \text{sign}(\eta) = (p+1,q)$$
$$T^*M^-_{C_{p,q}} = T^*M_{C_{p,q}} \text{ with } C < 0\ , \qquad \text{sign}(g) = (p,q)\,, \qquad \text{sign}(\eta) = (p,q+1)$$

where $\text{sign}(g)$ is the signature of the metric i.e.

$$\text{sign}(g) = (p,q) \Longrightarrow g = -\left(d\tilde{x}^1\right)^2 - \cdots - \left(d\tilde{x}^q\right)^2 + \left(d\tilde{x}^{q+1}\right)^2 + \cdots + \left(d\tilde{x}^{p+q}\right)^2$$

in some local coordinates $\tilde{x}^\mu$. For example $T^*M^-_{C_{1,3}}$ is dS and $T^*M^+_{C_{1,3}}$ is AdS.

(4) **Configuration space connection and curvature on the constant curvature manifold of codimension one-case** ($M_{C_{p,q}}$): We let ∂_a be the flat embedding connection of the ambient space and ∇_a to be the connection on the manifold $M_{C_{p,q}}$. Let f be an arbitrary function and let dx^μ be a basis of forms on the manifold $M_{C_{p,q}}$ then:

$$\nabla_\sigma f(x) = \frac{\partial f}{\partial x^\sigma}$$
$$\nabla_\sigma \left(dx^\mu\right) = -\Gamma^\mu_{\nu\sigma} dx^\nu$$
$$\nabla_\sigma \left(\frac{\partial}{\partial x^\mu}\right) = \Gamma^\nu_{\ \mu\sigma} \frac{\partial}{\partial x^\nu}$$
$$\nabla_{[\sigma}\nabla_{\rho]} \left(dx^\mu\right) = R^\mu_{\ \nu\sigma\rho} dx^\nu.$$

We can extend to higher-order tensors by using the Leibnitz rule and the fact that ∇ commutes with contractions.

(5) **Symmetrization and antisymmetrization of indices:**

$$2\Delta_{\mu\nu} = \Delta_{\mu\nu} - \Delta_{\nu\mu}$$
$$2\Delta_{(\mu\nu)} = \Delta_{\mu\nu} + \Delta_{\nu\mu}$$
$$n!\Delta_{(\mu_1\cdots\mu_n)} = \Delta_{\mu_1\cdots\mu_n} + \text{(all perms)}$$
$$n!\Delta_{[\mu_1\cdots\mu_n]} = \Delta_{\mu_1\cdots\mu_n} + \text{(even perms)} - \text{(odd perms)}\,.$$

(6) **Coordinates** and the corresponding **basis of one-forms** on phase–space where $\{q^A\} = (q^1, \ldots, q^{2n})$:

$$\left\{dq^1, \ldots, dq^{2n}\right\}.$$

(7) **Phase–Space connection:** Given an arbitrary function f and vector v^B on phase–space, a **general (torsion-free) phase–space connection:**

$$D_A f = \frac{\partial f}{\partial q^A}, \qquad D_A v^B = -\Gamma^A_{BC} v^C$$

with the conditions that D preserves the symplectic form $D \otimes \omega = 0$ and is torsion-free $D^2 f = 0$ (or in abstract indices: $D_A \omega_{BC} = 0$ and $D_{[A} D_{B]} f = 0$). We define the connection in the coordinates q^A as:

$$D \otimes \Theta^A = \Gamma^A_C \otimes \Theta^C = \Gamma^A_{CB} \Theta^B \otimes \Theta^C$$

and the curvature is:

$$D^2 \otimes \Theta^A := R_B^A \otimes \Theta^B = R^A_{CEB} \Theta^C \wedge \Theta^E \otimes \Theta^B.$$

*We note that these conditions do not specify D_A uniquely. We are free to add a tensor Δ_{ABC} symmetric in (ABC), i.e., a new connection D_{new} may be defined by:

$$D_{\text{new}} \otimes \Theta^A = \Gamma^A_{CB} \Theta^B \otimes \Theta^C + \Delta^A_{CB} \Theta^B \otimes \Theta^C.$$

Again, we can extend to higher-order tensors by using the Leibnitz rule and the fact that D commutes with contractions.

(8) **Flat connection:** When the phase–space is flat, i.e., associated to a flat space/space–time we will use ∂_A instead of D_A for the connection.

(9) **Antisymmetric and symmetric tensor products:** The wedge product $\wedge$ is reserved for the antisymmetric tensor product

$$\theta \wedge \alpha := \theta \otimes \alpha - \alpha \otimes \theta$$

and the vee product $\vee$ is reserved for the symmetric tensor product:

$$\theta \vee \alpha := \theta \otimes \alpha + \alpha \otimes \theta.$$

Since writing $\wedge$ and $\vee$ all over the place will become cumbersome we will make the convention that we will not write them because it will be obvious when we mean one or the other. For example, the metric always uses the symmetric tensor product $g = g_{\mu\nu} dx^\mu \vee dx^\nu$ and the symplectic form always uses the antisymmetric tensor product $\omega = \omega_{AB} \Theta^A \wedge \Theta^B$. However, we simply write them $g = g_{\mu\nu} dx^\mu dx^\nu$ and $\omega = \omega_{AB} \Theta^A \Theta^B$.

(10) Also, when we write D^2 or $(D - \hat{D})^2$ like in Eqs. (2.8) and (3.8) we always mean antisymmetric tensor products because these are curvature equations. In the curvature operators like D^2 or $(D - \hat{D})^2$ the Θ's are always wedged together by definition. An example is:

$$D^2 v_B = R_B{}^A v_A = R_{CEB}^{\;A} \Theta^C \wedge \Theta^E v_A.$$

If indices A, B, C, etc. are all abstract then the formula above is:

$$[D_A, D_B]\, v_B = 2R^A_{CEB} v_A.$$

(11) **The symplectic form:**

$$\omega = \omega_{AB} \Theta^A \wedge \Theta^B = \omega_{AB} \Theta^A \Theta^B$$

ω^{AB} is the inverse of ω_{AB} with $\omega^{AB} \omega_{BC} = \delta^A_C$.

(12) **The Poisson bracket:**

$$[f, g]_P = \omega^{AB} (D_A f)(D_B g)$$
$$\overleftrightarrow{P} := \overleftarrow{D}_A \omega^{AB} \overrightarrow{D}_B, \qquad f \overleftrightarrow{P} g = [f, g]_P$$

where f and g are two arbitrary functions and the arrows determine the direction that each derivative acts it.

(13) **Darboux coordinates and Darboux's Theorem:** In the neighborhood of each point on an n-dimensional symplectic manifold, there exists coordinates called **Darboux coordinates** $\tilde{q} = \left(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{p}_1, \ldots, \tilde{p}_n\right)$[11] where the ω takes the form:

$$\omega = d\tilde{p}_1 d\tilde{x}^1 + \cdots + d\tilde{p}_n d\tilde{x}^n.$$

(14) **Groenewold–Moyal star:** In terms of the flat connection ∂ Groenewold–Moyal star is:

$$\begin{aligned} f * g &= f \mathrm{e}^{\frac{i\hbar}{2} \overleftarrow{\partial}_A \omega^{AB} \overrightarrow{\partial}_B} g = fg + \frac{i\hbar}{2}\omega^{AB} (\partial_A f)(\partial_B g) - \frac{\hbar^2}{8}\omega^{CE}\omega^{AB} (\partial_C \partial_A f)(\partial_E \partial_B g) + \cdots \\ f * g &= \sum_{A,B,j}^{\infty} (i\hbar/2)^j\, \omega^{A_1 B_1} \cdots \omega^{A_j B_j} / j! (\partial_{A_1} \cdots \partial_{A_j} f)(\partial_{B_1} \cdots \partial_{B_j} g). \end{aligned} \tag{A.1}$$

(15) Smooth functions on a space A, $C^\infty(A)$.

(16) The **traces over translational degrees of freedom:**

$$\mathrm{Tr}_{tr}(\hat{f}) := \int \mathrm{d}^n x\, \langle x | \hat{f} | x \rangle$$
$$\mathrm{Tr}_{tr*}(f) := \frac{1}{(2\pi\hbar)^n} \int d^n p \mathrm{d}^n x\ f = \frac{1}{(2\pi\hbar)^n} \int \mathrm{d}^{2n} q\ f.$$

(17) The **traces over all degrees of freedom**, i.e., over the translation degrees of freedom as well as all other degrees of freedom is denoted by Tr and Tr_*.

(18) Let (N, ω) be a **symplectic manifold** where ω is a nondegenerate closed ($d\omega = 0$) two-form.

(19) **Formal series in** $\hbar$ is a power series in $\hbar$ with coefficients in A denoted by adding $[[\hbar]]$ like $A[[\hbar]]$.

For example, $C^\infty(T^*M)[[\hbar]]$ is formal series in $\hbar$ with coefficients in $C^\infty(T^*M)$. Let $f(q) \in C^\infty(T^*M)[[\hbar]]$ then

$$f(q) = f_j(q)\ \hbar^j = f_0(q) + f_1(q)\ \hbar + f_2(q)\ \hbar^2 + \cdots.$$

where $f_j(q) \in C^\infty(T^*M)$ for each j.

(20) **Star-exponential:**

$$\exp_*(f) := e_*^f := \sum_j (f*)^j / j! = 1 + f + \frac{1}{2!} f * f + \frac{1}{3!} f * f * f + \cdots$$

(21) **Complex-valued and Matrix-valued functions:** In this paper when we say f is a function/form we define it to be a complex Taylor series in its variables.[12] Explicitly:

$$f(u, \ldots, v) = \sum_{l,j} f_{j_1 \cdots j_l} u^{j_1} \cdots v^{j_l} \quad (j\text{'s are powers not indices})$$

where u and v are arbitrary.

So if f is a function/form of some subset or all of the quantities x, p, dx, dp, ω, $\hbar$ and i, it then commutes with the $\hat{y}$'s and will be called a complex-valued function/form. On the contrary a matrix-valued function/form is a complex Taylor series in $\hat{y}$ and possibly some subset or all of the quantities x, p, dx, dp, ω, $\hbar$ and i.

[11] Note that these $2n$ coordinates are different from the $2n + 2$ embedding coordinates (x^μ, p_μ).

[12] The set of all of these type of functions is sometimes called the enveloping algebra of its arguments.

So if $f(x, p, \mathrm{d}x, dp, \omega, \hbar, i)$ is a complex-valued function/form it then commutes with the $\hat{y}$'s. More explicitly with the matrix indices written (which are exceptions to our index conventions):

$$\left(\hat{y}^A\hat{y}^B\right)_{jk} = \sum_l \hat{y}^A_{jl}\hat{y}^B_{lk}$$
$$\left(\left[\hat{y}^A, f\right]\right)_{jk} := \hat{y}^A_{jk} f - f\hat{y}^A_{jk} = 0.$$

On the contrary a matrix-valued function/form does not. For this paper, we will not write these matrix indices explicitly.

Appendix B. The Proof of the integrability of (3.24)

We want to show that the condition (3.24) is integrable locally. Showing that following the P of the condition in (3.24) vanishes:

$$P\,((3.24)) = P\left(\left(P - dx^\mu\hat{\partial}_\mu\right) f^\nu_\sigma + \Gamma^\nu_{\rho\mu} dx^\mu f^\rho_\sigma - \Gamma^\nu_{\sigma\mu} dx^\mu + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta\right) dx^\sigma = 0 \tag{B.1}$$

where P is the differential operator:

$$P = \left(D + f^\sigma_\rho dx^\rho \hat{\partial}_\sigma\right)$$

where $\hat{\partial}_\mu := \partial/\partial s^\mu$ implies that the condition (3.24) is integrable locally by the Cauchy–Kovalevskaya theorem.

*Note that this is analogous to how Fedosov can locally integrate the solution for $\hat{D}$, i.e., by requiring that $\left(D - \hat{D}\right)^2 \hat{y}^A = 0$ in (2.8). However, before doing this by brute force we notice that D acting on everything in the equation above is just the configuration space connection ∇. Therefore, to simplify the calculation we will use abstract indices. The equation above in (B.1) (and in (3.24)) becomes the equation:

$$\left(\nabla_{[n} + f^d_{[n}\hat{\partial}_d\right)\left(R_{ca]}{}^b{}_m s^m + \left(\nabla_c + f^e_c\hat{\partial}_{|e|}\right) f^b_{a]}\right) = 0 \tag{B.2}$$

where P on configuration space quantities is $\left(\nabla_n + f^d_n\hat{\partial}_d\right)$.

First we note that we want f^a_b to be a globally defined object hence it should be made out of tensors. This rules out the trivial solution of $f^\sigma_\rho dx^\rho = -\Gamma^\sigma_{\rho\nu} s^\nu dx^\rho$.

Proof.

$$\begin{aligned}(\text{B.2}) &= \left(\nabla_{[n} + f^d_{[n}\hat{\partial}_d\right)\left(R_{ca]}{}^b{}_m s^m + \left(\nabla_c + f^e_c\hat{\partial}_{|e|}\right) f^b_{a]}\right)\\ &= \left(\nabla_{[n} R_{ca]}{}^b{}_m\right) s^m - R^b_{m[ac}\left(\nabla_{n]} + f^d_{n]}\hat{\partial}_d\right) s^m + \left(\nabla_{[n} + f^d_{[n}\hat{\partial}_d\right)\left(\nabla_c + f^e_c\hat{\partial}_{|e|}\right) f^b_{a]}.\end{aligned}$$

In abstract indices we have the identities:

$$Ds^a = 0$$
$$\nabla_a f(x, s) = \partial_a f - \Gamma^c_{ab} s^b \hat{\partial}_c f$$
$$\nabla_{[n}\nabla_{c]} f(x, s) = R^b_{enc} s^e \hat{\partial}_b f$$

and we have the second Bianchi identity:

$$\nabla_{[n} R_{ca]}{}^b{}_m = 0.$$

We also have the identity:

$$P^2 = D^2 + \frac{4}{3} p_\delta R^\delta{}_{(\beta\kappa)\gamma} f^\beta_\rho \mathrm{d}x^\kappa dx^\rho \left[s^\gamma, \cdot\right] + \left(P f^\sigma_\rho + \Gamma^\sigma_{\beta\kappa}\mathrm{d}x^\kappa f^\beta_\rho\right) dx^\rho \hat{\partial}_\sigma. \tag{B.3}$$

Proof of (B.3):

$$\begin{aligned}P^2 h &= \left(D + f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma}\right)^2 h \\ &= \left(D + f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma}\right)\left(D + f^{\psi}_{\kappa}\mathrm{d}x^{\kappa}\hat{\partial}_{\psi}\right) h \\ &= D^2 h + \underbrace{f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma}(Dh) + D\left(f^{\psi}_{\kappa}\mathrm{d}x^{\kappa}\hat{\partial}_{\psi} h\right)}_{(C)} + \underbrace{f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma}\left(f^{\psi}_{\kappa}\mathrm{d}x^{\kappa}\hat{\partial}_{\psi} h\right)}_{(E)}.\end{aligned} \tag{B.4}$$

We know that:

$$\begin{aligned}&\hat{\partial}_{\psi} h = \left[k_{\psi}, h\right]/i\hbar \\ &Dk_{\sigma} := -\frac{4}{3} R^{\delta}{}_{(\sigma\kappa)\beta}\mathrm{d}x^{\kappa} s^{\beta} p_{\delta} + \Gamma^{\rho}_{\sigma\kappa}\mathrm{d}x^{\kappa} k_{\rho}.\end{aligned}$$

We can easily prove that $D([f,h]) - [f, Dh] = [Df, h]$ for any matrix-valued functions $f(x,p,s,k)$ and $h(x,p,s,k)$ therefore:

$$i\hbar D\left(\hat{\partial}_{\psi} h\right) - i\hbar\hat{\partial}_{\psi}(Dh) = i\hbar\left[-\frac{4}{3}R^{\delta}{}_{(\psi\sigma)\beta} dx^{\sigma} s^{\beta} p_{\delta} + \Gamma^{\rho}_{\psi\sigma} dx^{\sigma} k_{\rho}, h\right]$$

and (C) becomes:

$$\begin{aligned}(C) &= i\hbar\left(f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma}(Dh) + D\left(f^{\sigma}_{\rho} dx^{\rho}\hat{\partial}_{\sigma} h\right)\right) \\ &= f^{\sigma}_{\rho} dx^{\rho}\left([k_{\sigma}, Dh] - D([k_{\sigma}, h])\right) + D\left(f^{\sigma}_{\rho} dx^{\rho}\right)[k_{\sigma}, h] \\ &= f^{\sigma}_{\rho} dx^{\rho}[Dk_{\sigma}, h] + D\left(f^{\sigma}_{\rho} dx^{\rho}\right)[k_{\sigma}, h] \\ &= f^{\psi}_{\rho} dx^{\rho}\left(-\frac{4}{3}R^{\delta}{}_{(\psi\kappa)\beta}\mathrm{d}x^{\kappa}\left[s^{\beta}, h\right] p_{\delta} + \Gamma^{\sigma}_{\psi\kappa}\mathrm{d}x^{\kappa}[k_{\sigma}, h]\right) + \left(Df^{\sigma}_{\rho}\right) dx^{\rho}[k_{\sigma}, h] \\ \Longrightarrow (C) &= \frac{4}{3} p_{\delta} R^{\delta}{}_{(\psi\kappa)\beta} f^{\psi}_{\rho}\mathrm{d}x^{\kappa} dx^{\rho}\left[s^{\beta}, h\right] + \left(Df^{\sigma}_{\rho} + f^{\psi}_{\rho}\Gamma^{\sigma}_{\psi\kappa}\mathrm{d}x^{\kappa}\right) dx^{\rho}\hat{\partial}_{\sigma} h\end{aligned}$$

also:

$$\begin{aligned}(E) &= f^{\sigma}_{\rho} dx^{\rho}\mathrm{d}x^{\kappa}\left(\hat{\partial}_{\sigma} f^{\psi}_{\kappa}\right)\hat{\partial}_{\psi} h + f^{\sigma}_{\rho} dx^{\rho} f^{\psi}_{\kappa}\mathrm{d}x^{\kappa}\left(\hat{\partial}_{\sigma}\hat{\partial}_{\psi} h\right) \\ &= f^{\sigma}_{\rho} dx^{\rho}\mathrm{d}x^{\kappa}\left(\hat{\partial}_{\sigma} f^{\psi}_{\kappa}\right)\hat{\partial}_{\psi} h.\end{aligned}$$

Putting (C) and (E) into the condition at (B.4):

$$\begin{aligned}&P^2 h = D^2 h + \frac{4}{3} p_{\delta} R^{\delta}{}_{(\psi\kappa)\beta} f^{\psi}_{\rho}\mathrm{d}x^{\kappa} dx^{\rho}\left[s^{\beta}, h\right] + \left(Pf^{\sigma}_{\rho} + f^{\psi}_{\rho}\mathrm{d}x^{\kappa}\Gamma^{\sigma}_{\psi\kappa}\right) dx^{\rho}\hat{\partial}_{\sigma} h \\ &\Longrightarrow P^2 = D^2 + \frac{4}{3} p_{\delta} R^{\delta}{}_{(\beta\kappa)\gamma} f^{\beta}_{\rho}\mathrm{d}x^{\kappa} dx^{\rho}\left[s^{\gamma}, \cdot\right] + \left(Pf^{\sigma}_{\rho} + \Gamma^{\sigma}_{\beta\kappa}\mathrm{d}x^{\kappa} f^{\beta}_{\rho}\right) dx^{\rho}\hat{\partial}_{\sigma}. \quad \square\end{aligned}$$

Using the identity in (B.3) in abstract indices gives:

$$\begin{aligned}&\left(\nabla_{[n} + f^{d}_{[n}\hat{\partial}_{d}\right)\left(\nabla_{c]} + f^{e}_{c]}\hat{\partial}_{e}\right) h \\ &\quad = \nabla_{[n}\nabla_{c]} h + \frac{2}{3} p_{d}\left(R^{d}{}_{(mc)a} f^{m}_{n} - R^{d}{}_{(mn)a} f^{m}_{c}\right)\left[s^{a}, h\right] + \left(\left(\nabla_{[n} + f^{d}_{[n}\hat{\partial}_{|d|}\right) f^{e}_{c]}\right)\hat{\partial}_{e} h \\ &\quad = \nabla_{[n}\nabla_{c]} h + \frac{2}{3} p_{d}\left(R^{d}{}_{(mc)a} f^{m}_{n} - R^{d}{}_{(mn)a} f^{m}_{c}\right)\left[s^{a}, h\right] + \left(\left(\nabla_{[n} f^{e}_{c]} + \left(\hat{\partial}_{d} f^{e}_{[c}\right) f^{d}_{n]}\right)\right)\hat{\partial}_{e} h\end{aligned}$$

where h is an arbitrary matrix-valued function of x and s.

Condition (B.2) becomes:

$$\begin{aligned}(\text{B.2}) &= -R^{b}_{\ m[ac}f^{m}_{\ n]} + \nabla_{[n}\nabla_{c}f^{b}_{\ a]} + \left(\nabla_{[n}f^{e}_{\ c} + \left(\hat{\partial}_{d}f^{e}_{\ [c}\right)f^{d}_{\ n}\right)\hat{\partial}_{|e|}f^{b}_{\ a]}\\
&= -R^{b}_{\ m[ac}f^{m}_{\ n]} + R^{d}_{\ [anc]}f^{b}_{\ d} - R^{b}_{\ d[nc}f^{d}_{\ a]} + \left(\hat{\partial}_{d}f^{b}_{\ [a}\right)R_{nc]\ e}^{\ \ \ d}s^{e} + \left(\nabla_{[n}f^{e}_{\ c} + \left(\hat{\partial}_{d}f^{e}_{\ [c}\right)f^{d}_{\ n}\right)\hat{\partial}_{|e|}f^{b}_{\ a]}\\
&= \left(\hat{\partial}_{d}f^{b}_{\ [a}\right)R_{nc]\ e}^{\ \ \ d}s^{e} + \left(\nabla_{[n}f^{d}_{\ c} + \left(\hat{\partial}_{e}f^{d}_{\ [c}\right)f^{e}_{\ n}\right)\hat{\partial}_{|d|}f^{b}_{\ a]}\\
&= \hat{\partial}_{d}f^{b}_{\ [a}\left(R_{nc]\ e}^{\ \ \ d}s^{e} + \nabla_{n}f^{d}_{\ c]} + \left(\hat{\partial}_{|e|}f^{d}_{\ c}\right)f^{e}_{\ n]}\right).\end{aligned}$$

Now according to the original condition (3.24) becomes the equation:

$$\nabla_{[n}f^{d}_{\ c]} + \left(\hat{\partial}_{e}f^{d}_{\ [c}\right)f^{e}_{\ n]} = -R^{d}_{\ enc}s^{e}$$

therefore:

$$(\text{B.2}) = \hat{\partial}_{d}f^{b}_{\ [a}\left(R_{nc]\ e}^{\ \ \ d}s^{e} - R^{d}_{\ |e|nc]}s^{e}\right) = 0$$

so modulo the original condition (3.24) the local integrability condition (B.2) is zero. Therefore $f^{b}_{\ a}$ exists at least locally by the Cauchy–Kovalevskaya theorem. □

Appendix C. Change of embedding

In this appendix we want to find $\hat{x}$ and $\hat{p}$ associated to the embedding:

$$x^{\mu}x_{\mu} = 1/C, \qquad x^{\mu}p_{\mu} = A.$$

To do this we exploit the canonical transformation:

$$\tilde{p}_{\mu} = p_{\mu} + CAx_{\mu}, \qquad \tilde{x}^{\mu} = x^{\mu}.$$

Since it leaves the symplectic form invariant:

$$\tilde{\omega} = d\tilde{p}_{\mu}d\tilde{x}^{\mu} = dp_{\mu}dx^{\mu} = \omega$$

as well as all other conditions except $x^{\mu}\tilde{p}_{\mu} = A$:

$$dx^{\mu}\tilde{p}_{\mu} + x^{\mu}d\tilde{p}_{\mu} = 0$$

it also leaves D and $\hat{D}$ unchanged. Therefore the two solutions:

$$\begin{aligned}\hat{x}^{\mu} &= \left(x^{\mu} + s^{\mu}\right)\frac{1}{\sqrt{Cu+1}}\\
\hat{p}_{\mu} &= \left(-Cz_{\nu}s^{\nu}x_{\mu} + z_{\mu}\right)\sqrt{Cu+1} - iC\hbar n\hat{x}_{\mu}\end{aligned}$$

are solutions still.

We perform the canonical transformation:

$$\begin{aligned}\hat{x}^{\mu} &= \left(x^{\mu} + s^{\mu}\right)\frac{1}{\sqrt{Cu+1}}\\
\hat{p}_{\mu} &= \left(-Cz_{\nu}s^{\nu}x_{\mu} + z_{\mu}\right)\sqrt{Cu+1} - iC\hbar n\hat{x}_{\mu}\end{aligned}$$

where:

$$p_{\mu} = \tilde{p}_{\mu} - CAx_{\mu}$$

and:

$$\widehat{\tilde{p}}_{\mu} = \hat{p}_{\mu} - CA\hat{x}_{\mu}$$

therefore:

$$\hat{x}^\mu = \left(x^\mu + s^\mu\right) \frac{1}{\sqrt{Cu+1}}$$
$$\widehat{\tilde{p}}_\mu = \left(z_\mu - C z_\nu s^\nu x_\mu\right) \sqrt{Cu+1} - C\,(i\hbar n + A)\,\hat{x}_\mu .$$

Appendix D. The derivation of the phase–space connection

Given the Levi-Civita connection ∇ on the configuration space M and subsequent curvature given the metric g on a general manifold M:

$$\begin{aligned} &\nabla_\sigma f(x) = \frac{\partial f}{\partial x^\sigma} \\ &\nabla_\sigma \left(dx^\mu\right) = -\Gamma^\mu_{\nu\sigma} dx^\nu \\ &\nabla_\sigma \left(\frac{\partial}{\partial x^\mu}\right) = \Gamma^\nu_{\mu\sigma} \frac{\partial}{\partial x^\nu} \\ &\nabla_{[\sigma} \nabla_{\rho]} \left(dx^\mu\right) = R^\mu{}_{\nu\sigma\rho} dx^\nu \end{aligned} \tag{D.1}$$

where $R^\mu{}_{\nu\sigma\rho}$ is the Riemann tensor. Of course we have the conditions that ∇ preserves the metric g and is torsion-free:

$$\nabla_a g_{bc} = 0$$
$$\nabla_{[a} \nabla_{b]} f(x) = 0$$

for all functions $f(x)$. Together these uniquely fix ∇ and give the standard formula for the Christoffel symbols:

$$\Gamma^\rho_{\mu\nu} = -\frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{D.2}$$

where ∂_μ are the partial derivatives in some basis x^μ. Define now a basis of covectors or forms $\Theta^B \in T^*T^*M$ (the cotangent bundle of the phase–space):

$$\Theta^B = \left(dx^\sigma, \alpha_\sigma\right)$$

where the dx's are the first $n\,\Theta$'s, the α's are the last $n\,\Theta$'s and they are defined to be:

$$\alpha_\mu := dp_\mu - \Gamma^\nu_{\mu\rho} dx^\rho p_\nu . \tag{D.3}$$

To extend D to define $D \otimes \alpha_\mu$ we require that D preserves the symplectic form ω:

$$0 = D \otimes \omega = D \otimes \left(\alpha_\mu dx^\mu\right) \Longrightarrow \left(D \otimes \alpha_\mu\right) dx^\mu = \left(\Gamma^\nu_{\mu\sigma} dx^\sigma \otimes \alpha_\nu\right) dx^\mu$$

where it can be shown that:

$$\omega = \alpha_\mu dx^\mu = dp_\mu dx^\mu \tag{D.4}$$

which can be proven by the torsion-free condition which tells us that $\Gamma^\nu_{[\mu\rho]} = 0$. Therefore we make the ansatz:

$$D \otimes \alpha_\mu := S_{\mu\rho\sigma} dx^\sigma \otimes dx^\rho + \Gamma^\nu_{\mu\sigma} dx^\sigma \otimes \alpha_\nu \tag{D.5}$$

where $S_{[\mu\rho]\sigma} = 0$.

We can fix $S_{\mu\rho\sigma}$ by requiring that the directional derivative $\mathcal{D}_v$ of a vector and covector in any direction v^a on the manifold is also a vector and covector respectively.

$$w_\mu \text{ is a covector } \iff \mathcal{D}_v w_\mu := v^\rho \left(\partial_\rho w_\mu - \Gamma^\nu_{\mu\rho} w_\nu\right) \text{ is a covector}$$
$$w^\mu \text{ is a vector } \iff \mathcal{D}_v w^\mu := v^\rho \left(\partial_\rho w^\mu + \Gamma^\mu_{\nu\rho} w^\nu\right) \text{ is a vector.}$$

This means that for any $p_\mu = w_\mu(x)$ (i.e., any section in the cotangent bundle) the directional derivative of a covector is a covector. Then the following formula must hold:

$$\nabla_{[a}\nabla_{b]}w_c = R^d{}_{cab}w_d$$

for every w_μ by the definition of the Riemann tensor. This formula then fixes the skew part of Eq. (D.5) to be:

$$D\alpha_\mu := D\wedge\alpha_\mu = S_{\mu\rho\sigma}dx^\sigma dx^\rho + \Gamma^\nu_{\mu\sigma}dx^\sigma\alpha_\nu = -R^\nu{}_{\mu\rho\sigma}p_\nu dx^\sigma dx^\rho + \Gamma^\nu_{\mu\sigma}dx^\sigma\alpha_\nu$$
$$\Longrightarrow S_{a[ce]} = -R^b{}_{ace}p_b.$$

Therefore we need to solve for S_{ace} that satisfies the two conditions:

$$S_{[ac]e} = 0 \;\&\; S_{a[ce]} = -R^b{}_{ace}p_b.$$

Let $S_{ace} := S^b{}_{ace}p_b$ and these conditions become:

$$S^b{}_{[ac]e} = 0 \;\&\; S^b{}_{a[ce]} = -R^b{}_{ace}. \tag{D.6}$$

Using the first Bianchi identity, the solution to this equation is:

$$S^b{}_{ace} = -\frac{4}{3}R^b{}_{(ac)e}. \tag{D.7}$$

Therefore:

$$D\otimes\alpha_\mu := -\frac{4}{3}R^\psi{}_{(\mu\sigma)\beta}p_\psi dx^\beta\otimes dx^\sigma + \Gamma^\nu_{\mu\sigma}dx^\sigma\otimes\alpha_\nu.$$

The phase–space connection is:

$$\begin{aligned}
&Dx^\mu := dx^\mu\\
&Dp_\mu := dp_\mu\\
&D\otimes dx^\mu = -\Gamma^\mu_{\sigma\nu}dx^\nu\otimes dx^\sigma\\
&D\otimes\alpha_\mu = \Theta^B\otimes D_B\alpha_\mu := -\frac{4}{3}R^\psi{}_{(\mu\sigma)\beta}p_\psi dx^\beta\otimes dx^\sigma + \Gamma^\nu_{\mu\sigma}dx^\sigma\otimes\alpha_\nu\\
&\alpha_\mu := dp_\mu - \Gamma^\nu_{\mu\rho}dx^\rho p_\nu
\end{aligned} \tag{D.8}$$

which is the connection in (3.7) and the corresponding curvature:

$$\begin{aligned}
&D^2x^\mu = 0\\
&D^2p_\mu = 0\\
&D^2\otimes dx^\mu = dx^\sigma dx^\rho\otimes R^\mu_{\nu\sigma\rho}dx^\nu\\
&D^2\otimes\alpha_\mu = \frac{4}{3}dx^\sigma\left(C^\psi_{\mu\beta\nu\sigma}p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma}\alpha_\nu\right)\otimes dx^\beta - R^\nu{}_{\mu\sigma\beta}dx^\sigma dx^\beta\otimes\alpha_\nu
\end{aligned} \tag{D.9}$$

which is the curvature in (3.8) where $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$ and according to (3.5) the formula for the curvature is:

$$R^\mu_{\nu\sigma\rho} = -\partial_{[\sigma}\Gamma^\mu_{\rho]\nu} + \Gamma^\kappa_{\nu[\sigma}\Gamma^\mu_{\rho]\kappa}. \tag{D.10}$$

We can extend to higher-order tensors by using the Leibnitz rule and the fact that D and ∇ commute with contractions.

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